Einführung in die Quantenoptik II Sommersemester 2008

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Problem set 5 Hand out: 04 June 2008
hand in: 17 June 2008

Problem 5.1 – Lindblad model for laser cooling (10 points)

In this problem, you re-derive friction and momentum diffusion for an atom within a simple master equation in Lindblad form. The model is based on Gallis (*Phys Rev A* 1993).

Consider a Lindblad master equation in the Heisenberg picture

$$\frac{\mathrm{d}\langle A\rangle}{\mathrm{d}t} = \mathrm{i}\langle [H,\, A]\rangle + \frac{1}{2}\sum_q \left\langle [L_q^\dagger,\, A] L_q + L_q^\dagger\, [A, L_q]\right\rangle \tag{5.1}$$

$$L_q = = \frac{A_q}{\sqrt{2}}\mathrm{e}^{\mathrm{i}qx}\left(1 - \tfrac{1}{2}\kappa q p\right) \tag{5.2}$$

where position x and momentum p are non-commuting system operators, the parameter q is running from $-\infty$ to ∞, the amplitudes A_q are specified later and the parameter κ also.

(i) Having in mind that the Lindblad operator L_q provides a quantum jump for the system state, try to give an interpretation of this model.

(ii) Compute the probability per unit time that any quantum jump occurs (summed over all possible jump operators L_q) and show that it is given by

$$\frac{1}{\tau} = \tfrac{1}{2}\sum_q |A_q|^2 \langle (1 - \tfrac{1}{2}\kappa q p)^2\rangle = \tfrac{1}{2}\sum_q |A_q|^2 \left(1 + \tfrac{1}{4}\kappa^2 q^2 \langle p^2\rangle\right) \tag{5.3}$$

(in the second expression, we have assumed that the coefficients $|A_q|$ are even in q)

(iii) Work out the equations of motion for the momentum operator and its square for a system Hamiltonian $H = p^2/2M$ (free particle),

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = -\alpha\langle p\rangle \tag{5.4}$$

$$\frac{\mathrm{d}\langle p^2\rangle}{\mathrm{d}t} = -\Gamma\langle p^2\rangle + 2D_p \tag{5.5}$$

and show that the damping and momentum diffusion coefficients are given by (no guarantee for factors 2 etc)

$$\alpha = \hbar\kappa \sum_q |A_q|^2 q^2 \tag{5.6}$$

$$\Gamma = 2\hbar\kappa \sum_q |A_q|^2 q^2 \left(1 - \tfrac{1}{4}\hbar\kappa q^2\right) \tag{5.7}$$

$$D_p = \frac{\hbar^2}{2}\sum_q |A_q|^2 q^2 \tag{5.8}$$

Summarize in words the meaning of the parameters κ and A_q that emerges from these results.

(iv) What regime is needed so that you get the Einstein relation

$$\frac{\langle p^2 \rangle_{\rm ss}}{m} = \frac{D_p}{m\alpha} \tag{5.9}$$

between kinetic temperature and friction? (5 bonus points) (Answer: $\alpha\tau \ll 1$.)

Problem 5.2 – Atom-light diffraction (10 points)

In the lecture, we have seen the following effective Hamiltonian for the interaction of a two-level atom with a standing light wave:

$$H_{\rm int} = ({\rm prefactor}) \frac{\hbar g^2}{2\Delta} \sigma_3 a^\dagger a \left(S_+^2 + S_-^2 \right) \tag{5.10}$$

where $S_\pm = \exp(\pm {\rm i} k_L x)$ are displacement operators for the quantized momentum. This operator is related to the dipole force as we shall see here.

(i) Compute for the classical standing wave field, $\mathbf{d}_{ge} \cdot \mathbf{E}(x) = -\frac{1}{2}\hbar\Omega\cos(k_L x)$, the dipole force from the formula found in the lecture:

$$\mathbf{F}_{\rm dip} = -\frac{\hbar\Delta}{4} \frac{\nabla|\Omega|^2}{\Delta^2 + \gamma^2 + |\Omega|^2/2} \tag{5.11}$$

and construct the potential $V_{\rm dip}(x)$ that generates this force. Take the large detuning limit and make the required identifications to recover the interaction Hamiltonian $H_{\rm int}$ of Eq.(5.10). In this way, you can fix prefactor and sign.

(ii) We have seen that the atomic momentum distribution after crossing the standing wave is given by (index $m/2$ is correct; error in the lecture)

$$f(m\hbar k_L) = \left| J_{m/2}(z) \right|^2, \qquad z = \frac{g^2 n t}{2\Delta}, \quad m = 0, \pm 2, \pm 4 \ldots \tag{5.12}$$

A plot of this function vs the "diffraction order" m is shown below. This function can be approximated by the following classical argument: the atomic beam is treated as a classical stream of particles. We make the approximation that the atoms cross the standing wave at $90°$ along a straight line. The momentum transfer is approximated as $p(x) = F_x(x)t$. By integrating over the possible values of x, you get the momentum distribution. Assume that the beam is uniform over one period of the standing wave and show that

$$f_{\rm cl}(p) = \frac{N}{\sqrt{p_{\max}^2 - p^2}}, \qquad |p| \le p_{\max} \tag{5.13}$$

where N is a normalization and $p_{\max}$ is proportional to the maximum gradient of the dipole potential. Compute $p_{\max}$ and verify the link $2z\hbar k_L = p_{\max}$ to the argument z of the Bessel functions.

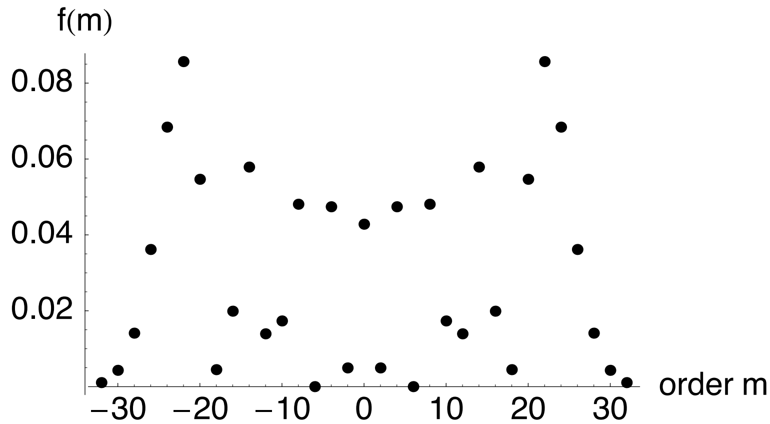

Figure 5.1: Distribution of discrete momentum transfers (in units of $\hbar k_L$) to an atom by a standing light wave. We have chosen $z = 13$.

(iii) In the quantum description, the diverging peaks of $f_{\text{cl}}(p)$ are thus broadened, there is a nonzero probability at $|p| > p_{\max}$. The oscillations are due to interference between classical trajectories that end up in the same momentum transfer. Make a sketch of these interfering trajectories.