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Problem set 6 Hand out: 17 June 2008
hand in: 01 July 2008

Problem 6.1 – Refractive index of a single atom (6 points)

What is the refractive index of a single atom that is put into a laser beam? In this problem, two routes are proposed to answer the question.

(i) Take the usual atom-field coupling Hamiltonian and argue that in the real part of $[H_{\text{int}}, a_k]$, linearized in a_k, the shift of the mode frequency $\delta\omega_k$ can be found. Argue that $\delta\omega_k \approx -(n(\omega_k) - 1)\omega_k$ for a small frequency shift.

Consider now the laser mode, $a_k = a_L$, and compute $\langle [H_{\text{int}}, a_L] \rangle$, in the stationary state of the two-level atom. Observe that the Rabi frequency Ω becomes an operator proportional to $\langle a_L \rangle$. You are allowed to make the approximation $\Omega \ll \gamma, \Delta$ to linearize the problem.

The "quantization volume" V still occurs in the index you find. Let us identify V with the "volume" of a cavity. Achim Peters (HU Berlin) has mentioned in a recent talk in Potsdam that frequency measurements with a relative precision of the order of 10^{-15} are possible. Find out how large the cavity can be to detect one atom at this level of precision.

(ii) Write the Maxwell wave equation for the electric field and express the current density $\mathbf{j}$ in terms of the polarization field $\mathbf{P}$. Consider a stationary situation (frequency $= \omega_L$) where the polarization is induced by the electric field itself: use the steady state average dipole from the lecture and translate it into an oscillating polarization density, given a number of atoms N per unit volume. Approximate, as in (i), the polarization by a linear function of the field amplitude and identify the dielectric function of the two-level medium. Compute the corresponding index of refraction. Estimate its amplitude $n - 1$ for one atom and sketch its behaviour as a function of the frequency detuning $\omega_L - \omega_A$.

Problem 6.2 – Electromagnetically induced transparency (8 points)

Consider a three-level atom in the "Λ-configuration" with the Hamiltonian

$$
\begin{aligned}
H &= \hbar\Delta_1 |1\rangle\langle 1| + \hbar\Delta_2 |2\rangle\langle 2| \\
&\quad + \frac{\hbar}{2}\left(\Omega_1 |e\rangle\langle 1| + \text{h.c.}\right) + \frac{\hbar}{2}\left(\Omega_2 |e\rangle\langle 2| + \text{h.c.}\right). \qquad (6.1)
\end{aligned}
$$

and subject to dissipative dynamics with the Lindblad operators

$$L_1 = \sqrt{\gamma}\,|1\rangle\langle e| \qquad (6.2)$$

$$L_2 = \sqrt{\gamma}\,|2\rangle\langle e| \qquad (6.3)$$

(i) Transform the Hamiltonian and the Lindblad operators into the basis $|s\rangle$, $|a\rangle$ defined by

$$\begin{aligned} |1\rangle &= \frac{\Omega_1 |s\rangle + \Omega_2^* |a\rangle}{\Omega} \\ |2\rangle &= \frac{\Omega_2 |s\rangle - \Omega_1^* |a\rangle}{\Omega}. \end{aligned}$$

where $\Omega = (|\Omega_1|^2 + |\Omega_2|^2)^{1/2}$. Show that the density matrix evolves according to the Bloch equations

$$\begin{aligned} \dot\rho_{ee} &= -2\gamma\,\rho_{ee} - \frac{\mathrm{i}}{2}\Omega\,(\rho_{se} - \rho_{es}) \\ \dot\rho_{es} &= (-\gamma + \mathrm{i}\Delta_1 d_s)\,\rho_{es} - \frac{\mathrm{i}}{2}\Omega\,(\rho_{ss} - \rho_{ee}) + \frac{\mathrm{i}}{2}\Delta_1 r^* \rho_{ea} \\ \dot\rho_{ea} &= (-\gamma + \mathrm{i}\Delta_1 d_a)\,\rho_{ea} - \frac{\mathrm{i}}{2}\Omega\rho_{sa} - \frac{\mathrm{i}}{2}\Delta_1 r \rho_{es} \\ \dot\rho_{ss} &= \gamma\,\rho_{ee} + \frac{\mathrm{i}}{2}\Omega\,(\rho_{se} - \rho_{es}) - \frac{\mathrm{i}}{2}\Delta_1\,(r^*\rho_{sa} - r\rho_{as}) \\ \dot\rho_{aa} &= \gamma\,\rho_{ee} + \frac{\mathrm{i}}{2}\Delta_1\,(r^*\rho_{sa} - r\rho_{as}) \\ \dot\rho_{sa} &= -\mathrm{i}\Delta_1\,(d_s - d_a)\,\rho_{12} - \frac{\mathrm{i}}{2}\Omega\rho_{ea} - \frac{\mathrm{i}}{2}\Delta_1 r\,(\rho_{aa} - \rho_{ss}) \end{aligned}$$

We have set here $\Delta_2 = 0$ and used the notations $d_a = |\Omega_2|^2/\Omega^2$, $d_s = |\Omega_1|^2/\Omega^2$, and $r = 2\Omega_1\Omega_2/\Omega^2$.

(ii) Show that for $\Delta_1 = 0$, the stationary state of the Bloch equations is $\rho_{aa} = 1$ and all other matrix elements being zero, as mentioned in the lecture.

(iii) Check that to first order in the detuning Δ_1, one gets

$$\begin{aligned} \rho_{ea} &\approx -\frac{\Delta_1 r}{\Omega} \\ \rho_{sa} &\approx -\frac{2\mathrm{i}\gamma\Delta_1 r}{\Omega^2} \end{aligned}$$

These are the expression that were needed in the lecture to find the group velocity in the three-level medium.

(iv) (5 bonus points) Find a numerical way to compute the steady state solution to the Bloch equations at arbitrary detuning. Plot the real and imaginary parts of the quantity

$$\frac{\Omega_1^*\rho_{e1}}{|\Omega_1|^2} = \frac{1}{\Omega}\rho_{es} + \frac{\Omega_1^*\Omega_2^*}{\Omega|\Omega_1|^2}\rho_{ea}$$

that plays the role of a refractive index.

Problem 6.3 – Adiabatic passage and Berry phase (6 points)

(a) Find the eigenstates of the Hamiltonian matrix

$$H = \begin{pmatrix} -\Delta & \Omega_1 & \Omega_2 \\ \Omega_1^* & 0 & 0 \\ \Omega_2^* & 0 & 0 \end{pmatrix}$$

and construct the matrix U whose column vectors are the eigenvectors. Observe that one eigenvalue is zero.

(b) Let the Rabi frequencies $\Omega_1(t)$ and $\Omega_2(t)$ change slowly in time. In this case, one may expect that the system is always in a "local eigenstate" (a column vector of $U(t)$). Writing the system state as

$$|\psi(t)\rangle = U(t)|\tilde{\psi}(t)\rangle,$$

the Schrödinger equation for $|\tilde{\psi}(t)\rangle$ reads (we have set $\hbar = 1$)

$$\mathrm{i}\frac{\partial}{\partial t}|\tilde{\psi}(t)\rangle = -\mathrm{i}U^\dagger(t)\frac{\partial U}{\partial t}|\tilde{\psi}(t)\rangle + U^\dagger(t)H(t)U(t)|\tilde{\psi}(t)\rangle$$

where the second matrix is diagonal (why?). The first matrix is small when the Rabi frequencies change slowly. To a first approximation, one may neglect its off-diagonal elements and is left with a Schrödinger equation where the three states are not coupled. However, the diagonal elements of the first matrix contribute to the phase of the states (the "Berry" or "geometric phase"). Compute this element for the eigenvector with zero eigenvalue.

(c) Discuss the Berry phase associated to the zero-eigenvalue state for your favorite choice of "pulses" $\Omega_1(t)$ and $\Omega_2(t)$ (laser fields slowly switched on and off, with possibly some phase sweep).