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**Introduction to Quantum Optics II**

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The preliminary programme for this lecture:

1. Reminder: Atom-field interaction, density matrices
2. Master equations in quantum optics
  - two-level atoms: spontaneous emission, Bloch equations, Glauber's photodetector theory, resonance fluorescence, quantum regression formula
  - positive maps, semigroups, the Lindblad master equation, entanglement
3. Applications
  - radiation forces, laser cooling, optical lattices
  - slow light (electromagnetically induced transparency)
  - quantum theory of the laser and the micromaser

# Chapter 0

## Atom-field interactions

### 0.1 Hamiltonian and relevant approximations

Hamiltonian  $H_A + H_F + H_{AF}$

electric dipole coupling in resonance (“rotating wave”) approximation

$$H_{AF} = -\hat{\mathbf{d}} \cdot \mathbf{E}(\mathbf{x}_A) \approx -\sum_k \mathcal{E}_k \left[ \mathbf{d}_{ge} \cdot \mathbf{f}_k(\mathbf{x}_A) \sigma^\dagger a_k + \text{h.c.} \right] \quad (1)$$

sum over field modes  $k$ , (classical) mode functions  $\mathbf{f}_k(\mathbf{x})$ , one-photon field amplitude  $\mathcal{E}_k = (\hbar\omega_k/2\varepsilon_0)^{1/2}$

two relevant atomic states  $|g\rangle$  and  $|e\rangle$ , transition matrix element of the electric dipole operator  $\mathbf{d}_{ge} = \langle g | \hat{\mathbf{d}} | e \rangle$ , two-level annihilation operator  $\sigma = |g\rangle\langle e|$

one-photon coupling frequency

$$\hbar g_k = -\mathcal{E}_k \mathbf{d}_{ge} \cdot \mathbf{f}_k(\mathbf{x}_A) \quad (2)$$

electric dipole approximation: atom size small compared to “relevant wavelengths”.

resonance approximation: field modes (incl. laser mode) near the resonance frequency  $\omega_A$ , hence only a narrow spectral band needed, with wavelength  $\lambda_A = 2\pi c/\omega_A \gg a_0$  (Bohr radius: size of atom)

“rotating-wave approximation”: only resonant (energy-conserving) terms in the interaction Hamiltonian,  $a_k \sigma^\dagger$  and  $a_k^\dagger \sigma$ : one photon disappears and atom becomes excited (absorption) or the inverse (photon emission).

## 0.2 Quantum states of a two-level system (“qbit”)

stationary states

pure state

$$|\psi(t)\rangle = \alpha(t)|e\rangle + \beta(t)|g\rangle \quad (3)$$

normalization  $|\alpha(t)|^2 + |\beta(t)|^2 = 1$  consistent with probability interpretation.

time evolution under “free atomic Hamiltonian” (in the Schrödinger picture)

$$\alpha(t) = \alpha(0) e^{-i\omega_A t/2} \quad (4)$$

$$\beta(t) = \beta(0) e^{i\omega_A t/2} \quad (5)$$

with our choice of zero energy. (Other choices: common, time-dependent phase factor.)

density matrix for a pure state  $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$  (projector), a hermitean  $2 \times 2$ -matrix

quantum average of atom observable  $\mathcal{A} = \sigma, \sigma_3, \dots$ :

$$\langle\mathcal{A}\rangle = \langle\psi|\mathcal{A}|\psi\rangle = \text{tr}[\mathcal{A}\rho] = \text{tr}[\rho\mathcal{A}] \quad (6)$$

mixed state: density matrix  $\rho$ , cannot be written as projector.

example: thermal state at temperature  $T$ ,

$$\rho = Z^{-1} (|g\rangle\langle g| + e^{-\hbar\omega_A/k_B T} |e\rangle\langle e|), \quad Z = 1 + e^{-\hbar\omega_A/k_B T} \quad (7)$$

Boltzmann factor for weighting the stationary states: combination of quantum and classical (ensemble) average.

geometric significance: Bloch vector and Bloch sphere

three components  $\mathbf{s} = (s_1, s_2, s_3)$ , expectation value of the spin operator

$$s_3 = \langle\sigma_3\rangle = |\alpha|^2 - |\beta|^2 \quad (8)$$

$$s_1 = \langle\sigma_1\rangle = \langle\sigma + \sigma^\dagger\rangle = 2 \text{Re}(\alpha^*\beta) \quad (9)$$

$$s_2 = \langle\sigma_2\rangle = (\cdot) \text{Im}(\alpha^*\beta) \quad (10)$$

sometimes complex notation used,  $s = \langle\sigma\rangle = (s_1 - i s_2)/2 = \alpha^*\beta$

pure state:  $1 = s^2 = s_1^2 + s_2^2 + s_3^2 = 4|s|^2 + s_3^2$ , on the surface of the Bloch sphere.

“north pole”: excited state  $|e\rangle$  (consistent with “spin up”)

“south pole”: ground state  $|g\rangle$  (“spin down”)

on the equator: superposition states with equal weight, e.g.  $(|g\rangle + e^{i\phi}|e\rangle)/\sqrt{2}$ . Relative phase  $\phi$  determines position on the equator:

$$\begin{aligned} s_1 &= \pm 1 \frac{|g\rangle \pm |e\rangle}{\sqrt{2}} \\ s_2 &= \pm 1 \frac{|g\rangle \pm i|e\rangle}{\sqrt{2}} \end{aligned} \quad (11)$$

(free) time evolution: rotation of the Bloch vector around the 3-axis with angular frequency  $\omega_A$ .

spin precession in the Heisenberg picture:

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{i}{\hbar} [H_A, \boldsymbol{\sigma}] = (\cdot)\omega_A \mathbf{e}_3 \times \boldsymbol{\sigma} \quad (12)$$

take expectation value: gives equation of motion for the Bloch vector.

rotation of the Bloch vector: pure states remain pure

mixed states: are located inside the Bloch sphere. For the thermal state (7), for example:

$$\mathbf{s} = -\mathbf{e}_3 \tanh(\hbar\omega_A/2k_B T) \quad (13)$$

in the “infinite temperature limit”,  $\mathbf{s} \rightarrow \mathbf{0}$ , the “completely mixed state”.

Quantitative measures of “being pure”: purity and entropy, see exercises.

# Chapter 1

## Master equations in quantum optics

### 1.1 Idea

A master equation describes the time evolution of a quantum system beyond the Schrödinger equation. It applies to “open systems” or systems for which the Hamiltonian is not completely known. Master equations are the quantum analogue of kinetic theories that describe the dynamics of a system including different dissipative effects. One can thus follow the approach of the system towards thermal equilibrium.

In quantum optics, the “system” can be an atom, a collection of atoms, or a field mode in a cavity. The system is “open” because it interacts with the “rest of the world”, manifest via a continuum of quantized field modes. An atom decays irreversibly by emitting a photon into previously empty vacuum modes. This phenomenon of “spontaneous emission” does not allow for a description in terms of a Hamiltonian. In fact, the state of the atom does not remain pure, and entropy increases because the photon can be emitted into any direction of space. A cavity mode decays because photons escape through the cavity mirrors, effectively becoming excitations of the “modes outside the cavity”. Alternatively, the field energy can be absorbed by the material making up the mirrors.

We focus first on the derivation of a master equation for a two-level atom. We shall find from the general theory two results: (1) the rate of

spontaneous emission of an electronically excited state and (2) Glauber’s formula for the signal of a photodetectors in terms of normally ordered correlation functions of the electric field operator.

### 1.1.1 Time scales

The derivation builds on the following hierarchy of time scales:

$$\frac{2\pi}{\omega_A} \lesssim \tau_c \ll \frac{2\pi}{\Omega} \sim \frac{1}{\gamma} \quad (1.1)$$

where the shortest time scale is the optical period of the light field that is near-resonant with the atomic transition frequency  $\omega_A$ . This is typically smaller than a few fs (femtoseconds) (in the visible range).

The “correlation time”  $\tau_c$  of the electromagnetic field describes qualitatively the time interval over which the quantum fluctuations of the electromagnetic field (that couple significantly to the atom) are “smooth”. Two electric field measurements are not significantly “similar” if they are taken at intervals larger than  $\tau_c$ . We give an estimate below, it turns out to be a few times larger than the optical period.

The third time scale is the Rabi period that scales with the inverse of the Rabi frequency of a laser. This depends of course on the laser power (and the atomic transition dipole), but typical values are in the 1–10 ns (nanoseconds) range. Note that this is much longer than the optical period and the field correlation time.

The last time scale is the lifetime of the excited state of the atom, inversely proportional to the spontaneous decay rate  $\gamma$ . This is typically in the 1–10 ns range as well, depending on the atom.

We thus have a situation similar to “Brownian motion”: the atom is “shaken around” by the vacuum field fluctuations that vary much faster than the typical dynamics for the atomic state (Rabi oscillations, spontaneous decay). The master equation takes advantage of this separation of time scales to find an equation of motion that can be applied on “slow time scales”. As a result, one gets a single equation that describes both, the “Hamiltonian” effects of a laser (Rabi oscillations) and the “dissipative / friction” effects due to vacuum fluctuations.

### 1.1.2 Field correlation time

To get an estimate of the field correlation time, we compute the two-time correlation (or coherence) function

$$C(\tau) = \langle E(t + \tau)E(t) \rangle \quad (1.2)$$

If we use the coupling constants  $g_k$  to write the mode expansion of the quantized field (this is equivalent to re-scaling the electric field so that it has units of frequency), we get

$$\begin{aligned} \langle E(t + \tau)E(t) \rangle &= \sum_k |g_k|^2 \langle a_k(t + \tau)a_k^\dagger(t) \rangle \\ &= \sum_k |g_k|^2 e^{-i\omega_k\tau} \end{aligned} \quad (1.3)$$

An explicit calculation using the mode amplitudes in free space leads to an integral of the form

$$C(\tau) \sim \int_0^\infty d\omega \omega^3 e^{-i\omega\tau} \quad (1.4)$$

which looks like a third derivative of a  $\delta$ -function in  $\tau$ . Well, not exactly since the integrations starts at  $\omega = 0$ .

Actually, we have forgot one thing in this estimate: the Hamiltonian we start with is only valid for field modes  $k$  near-resonant with the atomic transition,  $\omega_k \sim \omega_A$ . It thus seems reasonable to restrict the frequency integral in Eq.(1.4) to an interval around  $\omega_A$ , with a width  $\Delta\omega$  that is typically smaller than  $\omega_A$ .

Now, from the properties of the Fourier transformation, we know that the correlation function  $C(\tau)$  given by the integral (1.4) has a “width in time” given by  $\Delta\tau \Delta\omega \sim 1$ . Since this width is precisely the definition of the correlation time, we have

$$\frac{1}{\omega_A} \lesssim \frac{1}{\Delta\omega} \sim \tau_c. \quad (1.5)$$

Typical value: a few 10 fs.

## 1.2 Evolution of the atomic operators

We shall work in the Heisenberg picture. Note that operators describing different degrees of freedom (field, atom) commute at equal times.



### 1.2.1 Atomic dipole

“eliminate” the field operators by solving their equation of motion:

$$\frac{da_k}{dt} = \frac{i}{\hbar} [H, a_k] = -i\omega_k a_k - ig_k^* \sigma \quad (1.6)$$

$$a_k(t) = a_k(0) e^{-i\omega_k t} - ig_k^* \int_0^t dt' \sigma(t') e^{-i\omega_k(t-t')} \quad (1.7)$$

the “particular solution” contains the “past” of the atomic dipole operator  $\sigma(t')$ . Insert this into the equation for  $\sigma$ . Let’s look at this equation first:

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{i}{\hbar} [H, \sigma] = -i\omega_A \sigma + i \sum_k g_k [\sigma^\dagger, \sigma] a_k \\ &= -i\omega_A \sigma + i \sum_k g_k \sigma_3 a_k \end{aligned} \quad (1.8)$$

We see here that the two-level atom leads to nonlinear equations of motion: the operator product  $\sigma_3 a_k$  appears. For the moment, these operators commute (at equal times). But we now want to insert the solution (1.7) for  $a_k(t)$ , and the two terms that appear here do not commute separately with  $\sigma_3$ . For this reason, we take now a specific operator order (so-called “normal order”) where the annihilation operators ( $a_k$  or  $\sigma$ ) act first. This is the order already used in Eq.(1.8), and we thus get

$$\frac{d\sigma}{dt} = -i\omega_A \sigma + i\sigma_3 \mathcal{E}_0(t) + \int_0^t dt' \sum_k |g_k|^2 e^{-i\omega_k(t-t')} \sigma_3(t) \sigma(t') \quad (1.9)$$

where we used the abbreviation

$$\mathcal{E}_0(t) = \sum_k g_k a_k(0) e^{-i\omega_k t} \quad (1.10)$$

for the freely evolving electric field operator (positive frequency component only, and re-scaled by the atomic transition dipole). In the integral over  $t'$ , we see that the correlation function  $C(t - t')$  appears, and using  $\tau = t - t'$  as integration variable, we have

$$\frac{d\sigma}{dt} = -i\omega_A \sigma + i\sigma_3 \mathcal{E}_0(t) + \int_0^t d\tau C(\tau) \sigma_3(t) \sigma(t - \tau) \quad (1.11)$$

Now comes the *key observation*: under the time integral occur two very different functions. The correlation function  $C(\tau)$  is very narrow in  $\tau$ . The atomic operator  $\sigma(t - \tau)$  contains a “fast free evolution” (generated by the first term in its equation of motion), but once this is factored out, we anticipate that its evolution is “slow”:

$$\sigma(t - \tau) = e^{-i\omega_A(t-\tau)} \tilde{\sigma}(t - \tau) \approx e^{-i\omega_A(t-\tau)} \tilde{\sigma}(t) = e^{+i\omega_A\tau} \sigma(t) \quad (1.12)$$

Here, we introduced temporarily the dipole operator  $\tilde{\sigma}(t')$  (in an interaction picture).

The main benefit of this approximation is that only atomic operators at time  $t$  appear in the equation of motion. The two-level commutation rules<sup>1</sup> give  $\sigma_3(t)\sigma(t) = -\sigma(t)$ , and we end up with

$$\frac{d\sigma}{dt} = -i\omega_A\sigma + i\sigma_3\mathcal{E}_0(t) - \sigma(t) \int_0^t d\tau C(\tau) e^{i\omega_A\tau} \quad (1.13)$$

The upper integration limit  $t$  is actually the difference between our initial time and the actual time where the equation of motion is computed. We now let this time difference be much larger than the correlation time  $\tau_c$ . This is consistent with the assumption that the atomic dynamics is slow on the scale of the field’s correlation time. Then the integrand is effectively zero at the upper limit, and we get a constant complex number

$$\gamma + i\delta\omega_A = \int_0^{t \gg \tau_c} d\tau C(\tau) e^{i\omega_A\tau} = \frac{S(\omega_A)}{2} + i\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{S(\omega)}{\omega - \omega_A} \quad (1.14)$$

where  $S(\omega)$  is the Fourier transform of the correlation function  $C(\tau)$  and  $\mathcal{P}$  means the principal part of the integral.

Explicit calculation: spectrum of vacuum fluctuations

$$S(\omega) = 2\pi \sum_k |g_k|^2 \delta(\omega_k - \omega) \quad (1.15)$$

result from QO I:

$$S(\omega) = 2\pi \frac{1}{\hbar^2} (2/3) |\mathbf{d}_{ge}|^2 \frac{\hbar\omega}{2\varepsilon_0} \frac{4\pi\omega^2}{(2\pi c)^3} \quad (1.16)$$

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<sup>1</sup>A pedantic remark: operator products evolve as products under the Heisenberg equations of motion. This is because taking the commutator with a product is compatible with the product rule of (time) derivatives.

last factor: density of field modes per  $d\omega$  and volume. The spontaneously decaying atom is a “detector” for vacuum field fluctuations. The rate  $\gamma$  is also called the “natural linewidth” of the atomic transition  $|g\rangle \leftrightarrow |e\rangle$  because it gives the width in frequency of the spontaneous emission spectrum.

decay rate

$$\gamma = \frac{1}{2}S(\omega_A) = \frac{|\mathbf{d}_{ge}|^2(\omega_A/c)^3}{6\pi\hbar\epsilon_0} \quad (1.17)$$

order of magnitude, with  $|\mathbf{d}_{ge}| \sim ea_0$  (electron charge  $\times$  atom size)

$$\frac{\gamma}{\omega_A} \sim \alpha_{\text{fs}}(a_0/\lambda_A)^2 \quad (1.18)$$

with fine structure constant  $\alpha_{\text{fs}} = e^2/4\pi\epsilon_0\hbar c \approx 1/137.04$  and wavelength  $\lambda_A$  of resonant transition. Hence, indeed decay is “slow” on the scale of the optical period.

Frequency shift  $\delta\omega_A$  is related to asymmetry of vacuum spectrum around the transition frequency. Interpretation from second order perturbation theory: modes slightly below the atomic resonance,  $\omega_k \lesssim \omega_A$  tend to push the level  $|e\rangle$  upwards. Modes above resonance: push downwards. Null effect if spectrum is flat.

Actual calculation requires UV cutoff  $\omega_{\text{uv}}$  and interaction Hamiltonian beyond the resonance (rotating wave) approximation (Bethe, calculation of the Lamb shift). Order of magnitude:

$$\delta\omega_A \approx \gamma \log(\omega_{\text{uv}}/\omega_A) \quad (1.19)$$

and a consistent cutoff is the electron rest mass,  $\omega_{\text{uv}} \sim m_e c^2/\hbar$ . Hence comparable to the natural linewidth. Note: relativistic theory required, where wavelengths up to electron Compton wavelength  $\hbar/m_e c$  are involved. This goes beyond the “long wavelength approximation” because  $\hbar/m_e c \sim \alpha_{\text{fs}} a_0 \ll a_0$ .

## 1.2.2 Atomic populations

Actually, we did not yet show that  $\gamma$  is the rate of decay for the excited state. For this, we need the equation of motion for the occupations of the two energy levels. This is described by the atomic operator  $\sigma_3$ , also called

the “inversion” because  $\langle \sigma_3 \rangle > 0$  when the excited state is more occupied than the ground state.

Heisenberg equation of motion

$$\begin{aligned} \frac{d\sigma_3}{dt} &= +2i \sum_k [g_k^* a_k^\dagger \sigma - g_k \sigma^\dagger a_k] \\ &= 2i [\mathcal{E}_0^\dagger(t) \sigma - \sigma^\dagger \mathcal{E}_0(t)] \\ &\quad - 2 \int_0^t dt' [C^*(t-t') \sigma^\dagger(t') \sigma(t) + C(t-t') \sigma^\dagger(t') \sigma(t)] \end{aligned} \quad (1.20)$$

where we inserted the formal solution for  $a_k(t)$  and brought the operator products in normal order. We apply to the  $t'$ -integral the same prescription as before and get

$$\begin{aligned} &\int_0^t dt' [C^*(t-t') \sigma^\dagger(t') \sigma(t) + C(t-t') \sigma^\dagger(t') \sigma(t)] \\ &\approx (\gamma - i\delta\omega_A) \sigma^\dagger(t) \sigma(t) + (\gamma + i\delta\omega_A) \sigma^\dagger(t) \sigma(t) = \gamma(\sigma_3(t) + \mathbb{1}) \end{aligned} \quad (1.21)$$

where the frequency shift drops out. By construction, the operator  $\sigma_3(t) + \mathbb{1}$  gives the occupation of the excited state. From Eqs.(1.20, 1.21), we can thus read off the decay rate  $2\gamma$  for the excited state population, while the ground state remains stable.

Finally, the equation for the inversion operator becomes

$$\frac{d\sigma_3}{dt} = -2\gamma(\sigma_3(t) + \mathbb{1}) + 2i [\mathcal{E}_0^\dagger(t) \sigma - \sigma^\dagger \mathcal{E}_0(t)] \quad (1.22)$$

Combined with the equation of motion for the atomic dipole operator,

$$\frac{d\sigma}{dt} = -(\gamma + i\omega_A) \sigma + i\sigma_3 \mathcal{E}_0(t) \quad (1.23)$$

we have thus found the “optical Bloch equations”.

In Eq.(1.23), we have made the replacement  $\omega_A + \delta\omega_A \mapsto \omega_A$  for the atomic frequency. This is called “renormalization”: we combine the shift induced by the coupling to the vacuum field with the “naked” transition frequency into the frequency that can be physically observed. Recall that in reality, we can never “switch off” the coupling to the vacuum. Hence the “naked” two-level atom that we started with is actually a theoretical artefact.

### 1.3 Bloch equations with a coherent laser field

Note that we did not specify yet the state of the light field: it is in fact encoded in the operator  $\mathcal{E}_0(t)$  that depends on the initial field operators  $a_k(0)$ . Two examples will be studied now, the first one being an atom driven by a laser field. We have argued that to a good approximation, we can assume that the light field is in a coherent state  $|\alpha_L\rangle$ . We assume that at  $t = 0$ , the total system is in the product state  $|\psi(0), \alpha_L\rangle$  and take the expectation value of the Bloch equation. This gives the dynamics of the Bloch vector as follows (optical Bloch equations in the proper sense)

$$\frac{ds}{dt} = -(\gamma + i\omega_A)s + (i/2)s_3\Omega e^{-i\omega_L t} \quad (1.24)$$

$$\frac{ds_3}{dt} = -2\gamma(s_3(t) + 1) + i \left[ \Omega^*(t)e^{i\omega_L t}s - s^*\Omega e^{-i\omega_L t} \right] \quad (1.25)$$

where  $\Omega/2 = \langle \alpha_L | \mathcal{E}_0(0) | \alpha_L \rangle$  is the (complex) Rabi frequency and  $\omega_L$  the frequency of the laser mode.

These equations have time-dependent coefficients, but this can be removed by making a transformation into a “rotating frame”. We make the replacement

$$s(t) \mapsto s(t)e^{-i\omega_L t} \quad (1.26)$$

where the “new”  $s(t)$  satisfies Bloch equations with time-independent coefficients

$$\boxed{\begin{aligned} \frac{ds}{dt} &= -(\gamma - i\delta)s + (i/2)s_3\Omega \\ \frac{ds_3}{dt} &= -2\gamma(s_3(t) + 1) + i[\Omega^*s - s^*\Omega] \end{aligned}} \quad (1.27)$$

where  $\delta = \omega_L - \omega_A$  is the “laser detuning”. Note that the symbols are not the same throughout the books: the Rabi frequency  $\Omega$  or the decay rate  $\gamma$  can differ by a factor of 2 (or  $-2$ ), the detuning can have the opposite sign.

The Bloch equations are a “workhorse” of atomic physics and quantum optics. They are used to compute light absorption, excitation spectra, population transfer, radiation forces on atoms etc. In the exercises, you compute the stationary state of the Bloch equations (attention with the different sign for  $\delta$ : this one should be correct)

$$s_{ss} = \frac{-i(\Omega/2)(\gamma + i\delta)}{\gamma^2 + \delta^2 + \Omega^2/2} \quad (1.28)$$

$$s_{3,ss} = \frac{-(\gamma^2 + \delta^2)}{\gamma^2 + \delta^2 + \Omega^2/2} \quad (1.29)$$

Discussion: average atomic dipole operator (induced by laser field), average inversion. Line broadening.

Exercise: total excitation  $N$ , does not commute when laser included. Expectation value of  $\dot{N}$  in stationary state, interpretation as total scattered intensity.

Exercise: spectrum of spontaneous emission, from formal solution. Need formal solution of atomic dipole operator, Eq.(1.34).