Chapter 2

Master equations: axiomatic approach

Idea

The time evolution of an open quantum system can be understood as a mapping between density matrices. This mapping must satisfy some constraints, for example, it must preserve the probability interpretation of quantum mechanics. In addition, it seems reasonable that initial density matrixes that are "mixtures" of pure states evolve in a linear way and remain mixtures.

2.1 Positive and completely positive maps

We define a "dynamical map" $\Lambda : \rho \mapsto \Lambda(\rho)$ as a linear map of density matrices to density matrices. Actually, we only need "convex linearity" because this is the canonical way to generate mixed states:

$$\Lambda(\sum_k p_k \rho^{(k)}) = \sum_k \Lambda(\rho^{(k)}), \qquad p_k \geq 0, \sum_k p_k = 1 \tag{2.1}$$

but this construction is easily generalized to linear combinations with complex coefficients. The object $\Lambda(\rho)$ must qualify as a density matrix: hermitean, non-negative, and of trace unity.

In addition, we require "complete positivity" for a dynamical map: imagine that we enlarge the space on which ρ operates and extend Λ in

the following way to "larger" density matrices P. For factorized matrices, $\mathrm{P} = \rho \otimes \rho_{\mathrm{B}}$, we set

$$(\Lambda \otimes \mathbb{1})(\mathrm{P}) = \Lambda(\rho) \otimes \rho_{\mathrm{B}} \tag{2.2}$$

and extend this to arbitrary ("entangled") operators P by linearity. We then require that the extended map $(\Lambda \otimes \mathbb{1})$ is positive for any dimension of the extended space.

At first sight, complete positivity looks as a quite natural condition, not a very strong constraint. It reveals its full power as soon as non-factorized states P on the larger Hilbert space ("entangled states") enter the game.

2.2 Characterization of completely positive maps

It may come as a surprising fact that these conditions already imply a very special form for the dynamical maps: this is the

Kraus-Stinespring representation theorem: All dynamical maps are of the form

$$\Lambda(\rho) = \sum_k \Omega_k \rho \Omega_k^\dagger \tag{2.3}$$

with $\sum_k \hat{\Omega}_k^\dagger \hat{\Omega}_k = \mathbb{1}$.

Note that this equation generalizes the unitary evolution that we recover when the sum over the "Kraus operators" Ω_k contains only a single term. Exercise: Eq.(2.3) defines a completely positive map and preserves the trace of ρ.

Example. Imagine that you have a Hamiltonian $H(x)$ that depends on a "random parameter" x. It can take the values $x = x_k$ with probability p_k. This happens, for example, in your laboratory class when certain values of your apparatus are not well controlled. Then we can define the following "average density matrix" (denoted by the overbar) after time evolution under the unitary operator $U(x) = \exp[-\mathrm{i}H(x)t]$:

$$\rho \mapsto \overline{U(x)\rho U^\dagger(x)} = \sum_k p_k U(x_k)\rho U^\dagger(x_k) \tag{2.4}$$

Actually, from a quantum-mechanical perspective, this is the only way to describe the "preparation procedure" that you implement with the non-accurately known Hamiltonian. We observe that Eq.(2.4) is of the form of the Kraus theorem, with $\Omega_k = \sqrt{p_k}U(x_k)$.

Sketch of a proof, adapted from Chap. 4 in *Quantum Computing Devices: Principles, Designs and Analysis* by G. Chen & al, Taylor & Francis 2006, itself taken from Nielsen & Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press 2000).

You prove in the exercises that Eq.(2.3) defines a completely positive map. The only tricky point is the following extension to density operators P in a larger space

$$(\Lambda \otimes \mathbb{1})(\mathrm{P}) = \sum_k (\Omega_k \otimes \mathbb{1})\mathrm{P}(\Omega_k^\dagger \otimes \mathbb{1}) \tag{2.5}$$

where the factor $\otimes\mathbb{1}$ provides the necessary extension to larger dimensions.

We now want to prove the converse. Consider the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}$. Take a vector $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}$ and construct the operator

$$P = (\Lambda \otimes \mathbb{1})(|\phi\rangle\langle\phi|) \tag{2.6}$$

Since Λ is completely positive, and $|\phi\rangle\langle\phi|$ is a density operator, P is a (positive) density operator. Since P is hermitean, its spectral representation exists and can be written in the form

$$P = \sum_k |\tilde{\varphi}_k\rangle\langle\tilde{\varphi}_k|. \tag{2.7}$$

We have lumped the non-negative eigenvalues into the non-normalized eigenvectors $|\tilde{\varphi}_k\rangle$.

We now construct linear maps Ω_k on the system Hilbert space. Let $|\psi\rangle$, $|\chi\rangle \in \mathcal{H}$ and define its matrix elements as

$$\langle\chi|\Omega_k|\psi\rangle = (\langle\chi| \otimes \langle\psi^\star|)|\tilde{\varphi}_k\rangle \tag{2.8}$$

We use the notation $\langle\chi| \otimes \langle\psi^\star|$ for the tensor product between the bras (linear forms) $\langle\chi|$ and $\langle\psi^\star|$. The ket $|\psi^\star\rangle$ is defined with respect to a basis $\{|n\rangle\}$ of $\mathcal{H}$ by "taking the complex conjugate of the coefficients", i.e.:

$$\langle n|\psi^\star\rangle \equiv (\langle n|\psi\rangle)^* = \langle\psi|n\rangle \tag{2.9}$$

We can now check that for arbitary $|\chi\rangle, |\chi'\rangle, |\psi\rangle \in \mathcal{H}$, the following equalities hold

$$\begin{aligned}
&\sum_k \langle\chi|\Omega_k|\psi\rangle\langle\psi|\Omega_k^\dagger|\chi'\rangle \\
&= \sum_k (\langle\chi| \otimes \langle\psi^\star|)|\tilde{\varphi}_k\rangle\langle\tilde{\varphi}_k|(|\chi'\rangle \otimes |\psi^\star\rangle) \\
&= (\langle\chi| \otimes \langle\psi^\star|)P(|\chi'\rangle \otimes |\psi^\star\rangle) \qquad \text{from Eq.(2.7)} \\
&= (\langle\chi| \otimes \langle\psi^\star|)(\Lambda \otimes \mathbb{1})(|\phi\rangle\langle\phi|)(|\chi\rangle \otimes |\psi^\star\rangle) \qquad \text{from Eq.(2.6)}
\end{aligned} \tag{2.10}$$

We now specialize to the following form for the vector $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}$:

$$|\phi\rangle = \sum_n |n\rangle \otimes |n\rangle \tag{2.11}$$

(this vector is a so-called maximally entangled state on the product Hilbert space). Its projector admits the following expansion

$$\begin{aligned}
|\phi\rangle\langle\phi| &= \sum_{n,m} (|n\rangle \otimes |n\rangle)(\langle m| \otimes \langle m|) \\
&= \sum_{n,m} (|n\rangle\langle m|) \otimes (|n\rangle\langle m|)
\end{aligned} \tag{2.12}$$

in terms of skew operators $|n\rangle\langle m|$. It is quite astonishing that the full knowledge about Λ can be obtained by applying its extension (Eq.(2.6) to this single projector. We shall see in a moment that a dynamical map Λ (and its extensions) can be defined on skew operators as well, Eq.(2.15). Taking this for granted, we get

$$(\Lambda \otimes \mathbb{1})(|\phi\rangle\langle\phi|) = \sum_{n,m} \Lambda(|n\rangle\langle m|) \otimes (|n\rangle\langle m|) \tag{2.13}$$

Taking the matrix elements required in Eq.(2.10), and using the definition (2.9), we find

$$\begin{aligned}
\sum_k \langle\chi|\Omega_k|\psi\rangle\langle\psi|\Omega_k^\dagger|\chi'\rangle \qquad &= \sum_{n,m} (\langle\chi| \otimes \langle\psi^\star|) \left[\Lambda(|n\rangle\langle m|) \otimes (|n\rangle\langle m|)\right] (|\chi'\rangle \otimes |\psi^\star\rangle) \\
&= \sum_{n,m} \langle\chi|\Lambda(|n\rangle\langle m|)|\chi'\rangle\langle n|\psi\rangle\overline{\langle m|\psi\rangle} \\
&= \langle\chi|\Lambda(|\psi\rangle\langle\psi|)|\chi'\rangle.
\end{aligned} \tag{2.14}$$

In the last step, we have used the expansion of $|\psi\rangle$ in the basis $\{|n\rangle\}$. Hence, we have proven the operator identity (2.3) for the special case of a pure

state $\rho = |\psi\rangle\langle\psi|$. The proof is extended to a mixed state by decomposing ρ into projectors $|\psi_i\rangle\langle\psi_i|$ onto eigenvectors with non-negative weights (eigenvalues) p_i, and using the linearity of Λ.

To fill the gap, we need a prescription to apply a dynamical map to skew operators. We take

$$\begin{aligned} \Lambda(|\psi\rangle\langle\chi|) &= \frac{1}{2}\left[\Lambda(\rho_{+1}) - \Lambda(\rho_{-1}) + i\Lambda(\rho_{+i}) - i\Lambda(\rho_{-i})\right] \qquad (2.15) \\ \rho_u &:= \frac{1}{2}\left[(|\psi\rangle + u|\chi\rangle)(\langle\psi| + u^*\langle\chi|)\right], \qquad |u| = 1 \end{aligned}$$

where Λ is applied to projectors onto superposition states of ψ and χ with suitably chosen phase factors u. For a complex linear map, Eq.(2.15) is actually trivially satisfied, as a direct calculation shows. (See exercises.)

Remarks

- A map is completely positive if it is positive on the "doubled Hilbert space". This is actually all that we needed in the proof.
- If D is the dimension of the Hilbert space $\mathcal{H}$, then there are at most D^2 Kraus operators Ω_k. This is the maximum number of eigenvectors of P with nonzero eigenvalue (the maximum rank of P).
- The vector space of all linear maps is (at most) of dimension D^4: these "superoperators" can be written as $D^2 \times D^2$ matrices that act on the D^2-dimensional space of density matrices. (All dimensional estimates are actually upper limits here.) This suggests that the completely positive maps only cover a small subspace of all linear maps, in particular if D is large.
- The Kraus theorem provides us a characterization of all completely positive maps. Current research is turned towards a similar result for "positive maps". These maps, extended to the double Hilbert space, may image density operators onto operators with negative eigenvalues. This is connected to the generation of entanglement between the system and its "copy".

- There are researchers who do not accept the requirement of complete positivity (Pechukas, *Phys Rev Lett* 1994): they maintain that factorized states actually never occur in Nature (there are always some correlations or entanglement with the "rest of the world"). From this viewpoint, the violation of complete positivity is related to the fact that the (forgotten) correlations between the system and its environment are needed to construct the proper time evolution.

- Current research is aimed at extending or exploiting the Kraus theorem to master equations "with memory" (non-Markov case). At the time of writing, there are a few generalizations attempted, but no general result has been proven.

2.3 The Lindblad master equation

Semigroup with time-independent generator (Markov case): Lindblad form.

Semigroup: time evolutions $\rho(0) \mapsto \rho(t) = W(t)\rho(0)$ can be concatenated,

$$W(t_1 + t_2) = W(t_1)W(t_2) \tag{2.16}$$

what is missing from the usual group property: inverse element "$W(-t)$". Evolution is always "forward in time only" (related to dissipation and loss of information).

Eq.(2.16) is a "functional equation" that is formally solved by an operator of exponential form

$$W(t) = \exp(\mathcal{L}t) \tag{2.17}$$

where $\mathcal{L}$ is called the "generator" of the semigroup; it is itself time-independent. (The exponential map provides the homomorphism between concatenation of dynamical maps and addition of the time arguments.) Similar to the Kraus theorem, the constraints of linearity and complete positivity specify the structure of the generator. This is the so-called

Lindblad theorem. A completely positive semigroup $W(t) = \exp(\mathcal{L}t)$ has a generator $\mathcal{L}$ that implements the time evolution of a density operator ρ

in the form of the following differential equation. There is a hermitean operator H and a countable family of operators L (acting on the Hilbert space of the system) with

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\mathrm{i}\,[H,\,\rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\left\{ \rho, L_k^\dagger L_k \right\} \right) \\ &= -\mathrm{i}\,[H,\,\rho] + \frac{1}{2}\sum_k \left(\left[L_k \rho,\, L_k^\dagger \right] + \left[L_k,\, \rho L_k^\dagger \right] \right) \end{aligned} \tag{2.18}$$

The following sketch of a proof is adapted from Nielsen & Chuang and C. Henkel, *J Phys B* 2007.

We evaluate the difference quotient

$$\frac{\rho(t+\Delta t) - \rho(t)}{\Delta t} \tag{2.19}$$

with the help of the Kraus theorem and take the limit $\Delta t \to 0$. Write $\rho = \rho(t)$ for simplicity. In the Kraus representation (2.3) for the density matrix $\rho(t+\Delta t)$,

$$\rho(t+\Delta t) = \sum_k \Omega_k \rho \Omega_k^\dagger \tag{2.20}$$

the operators Ω_k depend on Δt. They can be split into

$$\Omega_k = \omega_k \mathbb{1} + V_k \tag{2.21}$$

where the V_k are uniquely defined by the requirement that their trace be zero. Now both ω_k and V_k depend on Δt.

In terms of these quantities, the change in the density matrix is computed to be

$$\begin{aligned} \rho(t+\Delta t) - \rho &= \left(\sum_k |\omega_k|^2 - 1 \right) \rho + \sum_k \left(\omega_k^* V_k\, \rho + \rho\, \omega_k V_k^\dagger \right) \\ &\quad + \sum_k V_k^\dagger \rho V_k \end{aligned} \tag{2.22}$$

where ω_k^* is complex conjugate to ω_k. We assume that the following continuity condition holds

$$\lim_{\Delta t \to 0} \left[\hat{A}\rho(t+\Delta t) - \hat{A}\rho(t) \right] = \mathcal{O}(\Delta t) \tag{2.23}$$

for all operators $\hat{A}$ and initial density matrices $\rho(t)$. This gives

$$\lim_{\Delta t\to 0}\sum_k |\omega_k|^2 = 1 \tag{2.24}$$

$$\lim_{\Delta t\to 0}\sum_k \omega_k^* \rho V_k \; = 0 \tag{2.25}$$

$$\lim_{\Delta t\to 0}\sum_k V_k\, \rho V_k^\dagger = 0 \tag{2.26}$$

where the last two lines apply to any density matrix ρ. We can thus introduce the derivatives

$$\gamma \;\equiv\; \lim_{\Delta t\to 0}\frac{\sum_k |\omega_k|^2 - 1}{\Delta t} \tag{2.27}$$

$$\Gamma - iH \;\equiv\; \lim_{\Delta t\to 0}\frac{\sum_k \omega_k^* V_k}{\Delta t} \tag{2.28}$$

where Γ and H are both hermitean.

Differentiating the condition that the dynamical map preserves the trace of the density matrix, we find

$$\begin{aligned} 0 \;&=\; \lim_{\Delta t\to 0}\frac{\mathrm{tr}\,[\rho(t+\Delta t)-\rho]}{\Delta t} \\ &=\; \mathrm{tr}\Big[\gamma\rho + 2\Gamma\rho + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\sum_k V_k^\dagger V_k\,\rho\Big] \end{aligned} \tag{2.29}$$

Since this must hold for any density matrix ρ, we find another derivative

$$\lim_{\Delta t\to 0}\frac{\sum_k V_k^\dagger V_k}{\Delta t} = -\gamma - 2\Gamma \tag{2.30}$$

We can thus introduce the Lindblad operators L_k by the limiting procedure

$$L_k \equiv \lim_{\Delta t\to 0}\frac{V_k}{\sqrt{\Delta t}} \tag{2.31}$$

Using the derivatives defined in Eqs.(2.27, 2.28, 2.31), we can divide the difference $\rho(t+\Delta t)-\rho(t)$ in Eq.(2.22) by Δt, and take the limit $\Delta t \to 0$. This gives the differential equation (2.18).

2.3.1 Examples

Spontaneous emission

is described by a single "Lindblad operator"

$$L = \sqrt{2\gamma}\sigma \tag{2.32}$$

where the strange unit arises because the "square of L" provides the actual time derivative of ρ.

We can best check that this is compatible with the Born-Markov master equation by switching to the Heisenberg picture. Taking the trace of the Lindblad master equation (2.18), multiplied with a system operator A, we find

$$\frac{\partial}{\partial t}\langle A\rangle = \mathrm{i}\langle[H,\,A]\rangle + \frac{1}{2}\sum_k \left\langle \left[L_k^\dagger,\,A\right] L_k + L_k^\dagger\,[A,\,L_k]\right\rangle \tag{2.33}$$

where the first term is the familiar one. The second one involves commutators between A and the Lindblad operators. Simple calculations show that this leads indeed to the damping of the atomic dipole operators σ and $\sigma^\dagger$ (at the rate γ, Eq.(1.23)) and to the damping of the inversion σ_3, Eq.(1.22), as we found in the previous chapter.

Rate equations in a thermal field

are generated by two Lindblad operators

$$L_{\mathrm{em}} = \sqrt{2\gamma(\bar{n}+1)}\,\sigma, \qquad L_{\mathrm{abs}} = \sqrt{2\gamma\bar{n}}\,\sigma^\dagger \tag{2.34}$$

where $\bar{n} = (\mathrm{e}^{\hbar\omega_{\mathrm{A}}/k_{\mathrm{B}}T} - 1)^{-1}$ is the average thermal photon number at the atomic transition frequency and temperature T of the radiation field. We recover the previous case for $T = 0$.

The operator L_{em} describes the emission of photons (spontaneous and stimulated) into the thermal field; the operator L_{abs} describes photon absorption. This can be easily checked by working out the equations of motion for the density matrix elements ρ_{gg} and ρ_{eg}. (One gets the rate equations that have been used by Einstein in his proof of the Planck spectrum (*Physik Zeitschr* 1917).)

Both rates add up in the dynamics of the atomic dipole (the matrix elements ρ_{eg}): their decay rate is $\gamma(2\bar{n}+1)$. This is a typical feature of master equations: the off-diagonal elements decay at least with the half-sum of the decay rates of the corresponding populations. In practice, their decay rate is even larger, due to additional dissipative processes ("dephasing").

Dephasing

is a process where only the off-diagonal elements of the density matrix decay, while the populations are left unchanged. The Lindblad operator is

$$L_{\mathrm{deph}} = \sqrt{\kappa/2}\,\sigma_3 \tag{2.35}$$

with a rate κ. By solving the Lindblad master equation (exercise!), we find

$$\rho(t) = \begin{pmatrix} \rho_{\mathrm{ee}}(0) & \mathrm{e}^{-\kappa t}\rho_{\mathrm{eg}}(0) \\ \mathrm{e}^{-\kappa t}\rho_{\mathrm{ge}}(0) & \rho_{\mathrm{gg}}(0) \end{pmatrix} \tag{2.36}$$

This process can be mimicked in a "classical way" by assuming that a superposition state vector

$$|\psi(t)\rangle = \alpha \mathrm{e}^{\mathrm{i}\varphi(t)}|\mathrm{e}\rangle + \beta \mathrm{e}^{-\mathrm{i}\varphi(t)}|\mathrm{g}\rangle \tag{2.37}$$

acquires a relative phase $\varphi(t)$ that is "randomly fluctuating". Experimentally, this happens for a two-level system embedded in a solid: the motion of the immediate environment perturbs the form of the electronic orbitals and hence their energy, even if the electron stays in this orbital ("adiabatic perturbation"). Hence only the energy is randomized, but the population is kept constant.

In this context, we can define a quantum-mechanical "average ensemble" by building the density matrix $|\psi(t)\rangle\langle\psi(t)|$ and taking the average over the probability distribution of $\varphi(t)$ (denoted by an overbar):

$$\rho(t) = \overline{|\psi(t)\rangle\langle\psi(t)|} \tag{2.38}$$

With the identification

$$\overline{\mathrm{e}^{\mathrm{i}\varphi(t)}} = \mathrm{e}^{-\kappa t} \tag{2.39}$$

we get the same result as with the Lindblad form. This is true if $\varphi(t)$ is a gaussian random variable with zero average and with variance $\langle\varphi(t)^2\rangle = \kappa t$. This behaviour is similar to Brownian motion (hence the name "phase diffusion"), in the mathematics literature, it is called a "Wiener process".

2.4 Exactly solvable open systems

Two examples for a two-level system coupled to a bath. One is based on "dephasing", the other one (see exercises) on "spontaneous emission". No

exact solutions are known when extra terms are added to the system Hamiltonian, for example, that break the simple form analyzed here.

2.4.1 Dephasing

References: N. G. van Kampen, *J Stat Phys* 1995 and G. Massimo Palma and Kalle-Antti Suominen and Artur K. Ekert, *Proc Roy Soc London A* 1996, in particular Section 4.

We consider a two-level system that couples to a quantized field (in the following: "bath") via

$$H_{\rm int} = \sigma_3 \sum_k \left(g_k b_k^\dagger + g_k^* b_k \right) \tag{2.40}$$

with coupling constants g_k that are summarized by the spectral density (ω_k is the frequency of bath mode k)

$$S(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k) \tag{2.41}$$

From the master equation (2.33) in the Heisenberg picture, we see that the inversion σ_3 is conserved. Hence, only the "off-diagonal operator" σ is affected by the bath. Going back to the Schrödinger picture, one can show that the off-diagonal elements of the density matrix behave like

$$\rho_{\rm eg}(t) = {\rm e}^{-\Gamma(t)} \rho_{\rm eg}(0) \tag{2.42}$$

where the "decoherence factor" is given by

$$\Gamma(t) = \frac{1}{2} \sum_k |\xi_k(t)|^2 \coth(\beta\omega_k/2) \tag{2.43}$$

where $1/\beta$ is the temperature of the initial bath state (we assume factorized initial conditions) and

$$\xi_k(t) = 2g_k \frac{1 - {\rm e}^{{\rm i}\omega_k t}}{\omega_k} \tag{2.44}$$

Discussion

For short times, we can expand the effective coupling constants $\xi_k(t)$ and get

$$t \to 0: \qquad \Gamma(t) \approx 2t^2 \sum_k |g_k(t)| \coth(\beta\omega_k/2) = 2t^2 \int_0^\infty \mathrm{d}\omega\, S(\omega) \coth(\beta\omega/2) \tag{2.45}$$

The quadratic dependence on time is characteristic for this initial regime. In fact, from perturbation theory, we see that the probability amplitude for states orthogonal to the initial one must increase linearly in t. The corresponding probability thus starts off proportional to t^2. The integral in Eq.(2.45) is often dominated by large frequencies, of the order of the "UV cutoff" $1/\tau_\mathrm{c}$. (Without this cutoff, the integral actually diverges and the short-time regime may even lead to mathematical inconsistencies.) The quadratic regime then applies only on time scales $t < \tau_\mathrm{c}$ that are typically very short compared to the dissipative dynamics.

At larger times, we can make the approximation that $|\xi_k(t)|^2$ approaches a δ-function:

$$t \to \infty: \qquad \left| \frac{1 - \mathrm{e}^{\mathrm{i}\omega_k t}}{\omega_k} \right|^2 \to 2\pi t\, \delta^{(t)}(\omega_k) \tag{2.46}$$

where the width of the δ-function is of the order $1/t$. In this limit, only low-frequency modes contribute to the decoherence factor.

Let us first assume that $1/t$ is larger than $1/\beta$ (intermediate range $\tau_\mathrm{c} \ll t \ll \hbar/k_\mathrm{B}T$). Then we can make the zero-temperature approximation $\coth(\beta\omega/2) \approx 1$ for the relevant modes and get

$$\tau_\mathrm{c} \ll t \ll \hbar/k_\mathrm{B}T: \qquad \Gamma(t) \approx 4\pi S(0) t \tag{2.47}$$

hence an exponential decay with a rate $\kappa = 4\pi S(0)$ that involves the spectral strength at zero frequency (more precisely: at frequencies $\omega \geq T/\hbar \ll \omega_\mathrm{c}$). This behaviour is also what we have found within the simple Lindblad master equation. (Recall: the master equation is not valid on the short time scale τ_c that sets the correlation time of the bath fluctuations.)

Finally, when $t \gg \hbar/k_\mathrm{B}T$, we have to take into account the thermal occupation of the low-frequency modes. The integral cannot be performed any more without knowledge of the behaviour of the function $S(\omega)$. Typical results are power laws $\mathrm{e}^{-\Gamma(t)} \propto t^\alpha$ with exponents α that depend on

$S(\omega)$ and the temperature. An exponential decay at a T-dependent rate is possible as well.

2.4.2 Spontaneous decay

This model is studied in the exercises. A two-level atom is coupled to a bosonic environment within the rotating-wave approximation,

$$H_{\rm int} = \sum_k \left(g_k \sigma^\dagger a_k + g_k^* a_k^\dagger \sigma \right) \tag{2.48}$$

Since the quantum number "excitation" (see exercises) is conserved, the subspace spanned by the states by $|{\rm e}, {\rm vac}\rangle$ and $|{\rm g}, 1_k\rangle$ (atom in ground state and one photon in mode k) is closed under time evolution. One can find a closed integro-differential equation for the amplitude $c_{\rm e} = \langle {\rm e}, {\rm vac}|\psi(t)\rangle$ of the state vector and solve it with the Laplace transform. The result is a non-exponential decay. If the spectral strength of the bath contains sharp peaks, the decay may even happen in an oscillating manner. Mathematically, this emerges from different poles in the Laplace transform of $c_{\rm e}(t)$ that give interfering contributions in the backtransformation. A simple exponential decay emerges at long times when one single pole is located close to the imaginary axis (in the Laplace variable).

Another generic feature are algebraic (non-exponential) "tails" that survive at long times whenever the spectral strength $S(\omega)$ can only be defined in a cut complex plane. This typically happens due to a discontinuity in some derivative at zero frequency. As a consequence, at very long times, the decay is not exponential any more. This feature has not yet been measured, to our knowledge. It may actually be an artefact of the factorized initial conditions for this model. In fact, if similar techniques are applied for the Bloch equations in this context, one can easily generate solutions that "leave the Bloch sphere", i.e., with negative eigenvalues of the density matrix. This is manifestly non-physical, but it does not seem obvious how to repair this problem. See, e.g., Davidson & Kozak, *J Math Phys* 1971 and Barnett & Stenholm, *Phys Rev A* 2001.

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