Einführung in die Quantenoptik

Sommersemester 2009
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Problem 3.1 – Projekt Wikipedia (2 points)
[Abgabe per email bis zum 02. Juni 2009]

Melden Sie die Hörerschaft der Vorlesung kollektiv als Wikipedia-Autor an. Erkunden Sie die Syntax für einen Eintrag: eine html-Variante? wie werden Formeln eingegeben?

Problem 3.2 – Transverse δ-function (7 points)
In the lecture, we have encountered the so-called 'transverse δ-function' $\delta^{\perp}_{ij}(\mathbf{x})$. It has the following properties ($\mathbf{F}(\mathbf{x})$ is an arbitrary vector field)

$$\int \mathrm{d}^3x' \sum_j \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}')F_j(\mathbf{x}') = \begin{cases} F_i(\mathbf{x}) & \text{if } \mathbf{F}(\mathbf{x}) \text{ is transverse} \\ 0 & \text{if } \mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x}) \end{cases} \tag{3.1}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}(\delta_{ij} - \frac{k_i k_j}{k^2}) \tag{3.2}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \delta_{ij}\delta(\mathbf{x}) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\mathbf{x}|} \tag{3.3}$$

(1) Show that Eq.(3.2) correctly implements property (3.1) by working with the spatial Fourier transform of the vector function $\mathbf{F}(\mathbf{x})$.

(2) Prove Eq.(3.3) by analogy to a gauge transformation: Be $\mathbf{F}(\mathbf{x})$ any (sufficiently smooth) vector field. Find a 'gauge function' $\chi(\mathbf{x})$ such that $\mathbf{F}'(\mathbf{x}) := \mathbf{F}(\mathbf{x}) + \nabla\chi(\mathbf{x})$ is transverse.

Hint. The 'gauge function' is the solution to an inhomogeneous Laplace equation.

Problem 3.3 – Commutators of free fields (6 points)
In the lecture, we have found that the commutator between the quantized vector potential $\mathbf{A}$ and its conjugate momentum field $\mathbf{\Pi}$ should be

$$[A_i(\mathbf{x}), \Pi_j(\mathbf{x}')] = \mathrm{i}\hbar\delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}') \tag{3.4}$$

(i) Take the rotation with respect to $\mathbf{x}$ and the index i and observe that this is nonzero. Show that one gets the so-called Pauli commutator between the (transverse) fields $\mathbf{E}$ and $\mathbf{B}$:

$$[B_i(\mathbf{x}), E_j(\mathbf{x}')] = \frac{\mathrm{i}\hbar}{\varepsilon_0}\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{x}-\mathbf{x}') \tag{3.5}$$

(ii) In a mathematically careful (quantum) field theory, the fields $\mathbf{A}$, $\mathbf{E}$, $\mathbf{B}$ etc are actually (operator-valued) distributions that have to be 'smeared out' with suitable smooth test functions $\mathbf{f}(\mathbf{x})$, $\mathrm{g}(\mathbf{x})$. This gives operators that behave in a less singular way; in particular you can multiply fields evaluated at the same point. Consider thus the observables

$$\mathcal{E}[\mathbf{f}] = \int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}), \qquad \mathcal{B}[\mathbf{g}] = \int \mathrm{d}^3x\, \mathbf{g}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \tag{3.6}$$

Work out the commutator $[\mathcal{E}[\mathbf{f}], \mathcal{B}[\mathbf{g}]]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{B}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E}[\mathbf{f}])^2$, $(\Delta\mathcal{B}[\mathbf{g}])^2$.

(iii) [4 bonus points] In the construction of (iii), focus on mode functions of the form $\nabla \times \mathrm{g} = (\omega/c)\mathbf{f}$ with $\mathbf{f}$ being normalized as $\int \mathrm{d}^3x\, \mathbf{f}^2(\mathbf{x}) = 1$ and $\omega > 0$ a constant. Consider a quantum state for the field where the average values of $\mathcal{E}[\mathbf{f}]$, $\mathcal{B}[\mathbf{g}]$ are zero and their variances are identical. What do you get then for the 'energy' $\varepsilon_0 \langle \mathcal{E}[\mathbf{f}]^2 \rangle + (1/\mu_0) \langle \mathcal{B}[\mathbf{g}]^2 \rangle$?

Answer: One 'photon energy' $\hbar\omega$ spread over the spatial size of the test function $\mathbf{f}(\mathbf{x})$.

Problem 3.4 – Orders of magnitude (6 points)

(i) The energy density of the quantized electromagnetic field, at temperature T is given by

$$u(T) := \langle u \rangle_T = \sum_k \frac{2\hbar\omega_k}{V} (\bar{n}(\omega_k) + \tfrac{1}{2}) \tag{3.7}$$

where the average photon number is given by the Bose-Einstein distribution $\bar{n}(\omega) = 1/(\mathrm{e}^{\hbar\omega/k_B T} - 1)$. Take the 'continuum limit' $V \to \infty$ where the sum over $\mathbf{k}$ is replaced by an integral to get the Planck-formula.

Focus on $T = 0$, switch to an integral over the frequency ω and speculate about a reasonable value for an upper limit frequency ('cutoff'), e.g., the largest mass of a known elementary particle or the Planck energy (see Wikipedia). Find a numerical estimate for the 'vacuum energy density' and check that for a cutoff at the Planck energy, it is about a factor $10^{100\ldots120}$ larger than the density of dark energy (about the critical mass density in the Universe, $\sim 10^{-27}\mathrm{kg/m}^3$ times c^2). What cutoff frequency would give exactly this energy density? Is that a reasonable cutoff?

(ii) Estimate the electromagnetic energy density (per frequency) you would get from the Planck formula (blackbody radiation) for microwave photons in the rectangular box of a microwave oven at a temperature of $T = 100°\,\mathrm{C}$. Compare to the actual value of the energy density when the oven is operated at $1000\,\mathrm{W}$, say. You may use that the microwaves are absorbed by the damped rotational motion of water molecules, whose damping rate is not very much smaller than the microwave frequency.