Einführung in die Quantenoptik

Sommersemester 2009
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Übungsaufgaben Blatt 6

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Problem 6.1 – A 'non-Markovian' refinement of spontaneous decay (6 points)

(i) In the lecture, we have discussed the master equation for a two-level system coupled to a radiation field with a short correlation time. The approximation we have used is called the 'Markov approximation' – this means that (i) the system observables (spin operator) at time t are sufficient to compute the time derivative and (ii) the terms in the master equation are not (explicitly) time-dependent (except for a possibly time-dependent atom-laser interaction). The Markov approximation neglects the 'memory time' of the radiation field.

A simple generalization of this approach is to define a time-dependent decay rate by keeping the upper integration limit t (that was approximated by $\tau_c \ll t \to \infty$ in the Markov approximation):

$$\gamma(t) = \mathrm{Re} \int_0^t \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}\omega_A \tau} C(\tau) \tag{6.1}$$

where $C(\tau)$ is the correlation function of the field and ω_A the Bohr frequency of the two-level atom.

(i) Show that the rate $\gamma(t)$ can become negative for a suitable choice of the spectrum $S(\omega)$.

(ii) Solve the non-Markovian equation for spontaneous decay ($s = \langle \sigma \rangle$ is the average dipole operator)

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -[\mathrm{i}\omega_A(t) + \gamma(t)]s \tag{6.2}$$

where the Bohr frequency $\omega_A(t)$ is time-dependent as well (by the same argument as $\gamma(t)$). Show that $|s(t)| \le |s(0)|$ and that $s(t \to \infty) = 0$, hence we get a strict decay at all times. Prove first that

$$\int_0^t \mathrm{d}t' \, \gamma(t') = 2 \int \frac{\mathrm{d}\omega}{2\pi} S(\omega) \, \mathrm{sinc}^2[(\omega - \omega_A)t] \ge 0 \tag{6.3}$$

where $S(\omega) \ge 0$ is the spectrum of the radiation field and $\mathrm{sinc}\, x = (\sin x)/x$.

Problem 6.2 – Jaynes–Cummings–Paul model (6 points)
In the lecture, we have needed the 'reduced dynamics' of the field mode after resonant coupling during a time τ with an initially excited two-level atom. Show that the mode's density operator ρ is given by

$$\begin{aligned}\rho(t+\tau) &= \cos(g\tau\sqrt{n+1})\rho\cos(g\tau\sqrt{n+1}) \\ &\quad + a^\dagger\frac{\sin(g\tau\sqrt{n+1})}{\sqrt{n+1}}\rho\frac{\sin(g\tau\sqrt{n+1})}{\sqrt{n+1}}a \end{aligned} \tag{6.4}$$

where n is the photon number operator (whose square root is defined by its action on the eigenvectors). You may want to solve first for the state of the atom+field mode system with initial condition $|\mathrm{e}, n\rangle$.

Problem 6.3 – Micromaser (6 points)
(i) Take matrix elements of the reduced dynamics of Eq.(6.4) in the Fock basis and average with respect to an exponential distribution of interaction times τ (average value $\bar{\tau}$). Show that one gets for the diagonal elements (the overbar denotes the average)

$$\overline{\rho_{nn}(t+\tau)} - \rho_{nn}(t) = -\frac{2(g\bar{\tau})^2(n+1)}{1+4(g\bar{\tau})^2(n+1)}\rho_{nn} + \frac{2(g\bar{\tau})^2 n}{1+4(g\bar{\tau})^2 n}\rho_{n-1n-1} \tag{6.5}$$

Argue that the same calculation can be done directly for the gain function $G(n) = r\sin^2(g\tau\sqrt{n})$ that we have identified from the master equation in the lecture. Show in particular that the micromaser threshold is given by $2r(g\bar{\tau})^2 \geq \kappa$.

(ii) When the interaction time τ is precisely controlled, the micromaser can be 'trapped' in highly non-classical states. These 'trapping states' can be found from the recurrence relation $p_n = [G(n)/\kappa n]p_{n-1}$, when $G(n) = 0$ for some n. Calculate the photon statistics p_n under this condition and sketch in words the time evolution of the micromaser from an initial vacuum state $\rho(0) = |0\rangle\langle 0|$.