Chapter 2

The quantized field

We give an elementary introduction to the quantization of the electromagnetic field. We adopt the Coulomb gauge and a simple canonical framework. We then discuss examples for the quantum states of the field and the properties of vacuum fluctuations.

2.1 Canonical quantization

2.1.1 Fields

Maxwell equations. We want to quantize the vacuum Maxwell equations for the electromagnetic field. With a given charge and current density, these read

$$\begin{aligned} \nabla \cdot \mathbf{B} = 0 \qquad & \varepsilon_0 \nabla \cdot \mathbf{E} = \rho \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \qquad & \nabla \times \mathbf{B} - \frac{1}{c^2} \partial_t \mathbf{E} = \mu_0 \mathbf{j} \end{aligned} \tag{2.1}$$

Introducing the scalar and vector potentials via

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.2}$$

$$\mathbf{E} = -\nabla \phi - \partial_t \mathbf{A}, \tag{2.3}$$

the left column of (2.1) is identically fulfilled. The Coulomb law then becomes

$$-\varepsilon_0 \triangle \phi - \varepsilon_0 \nabla \partial_t \mathbf{A} = \rho \tag{2.4}$$

If we impose the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, the vector potential drops out and the scalar potential is determined by the charge density alone:

$$-\varepsilon_0 \triangle \phi = \rho \tag{2.5}$$

In this gauge, the scalar potential is thus not a dynamical degree of freedom of the field: its dynamics is 'enslaved' by that of the charges. This holds with suitable boundary conditions such that the homogeneous equation $\triangle\phi = 0$ has no nontrivial solutions. In free space, with $\phi(\mathbf{x} \to \infty) \to 0$, we get

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \mathrm{d}^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{2.6}$$

which is a superposition of well-known Coulomb potentials.

Wave equation. We are left with the wave equation for the vector potential

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\partial_t^2 \mathbf{A} = \mu_0 \left(\mathbf{j} - \varepsilon_0 \partial_t \nabla\phi\right) \equiv \mu_0 \mathbf{j}_\perp. \tag{2.7}$$

On the right hand side, we have introduced the 'transverse current'. Its divergence is zero because of Eq.(2.5) and charge conservation:

$$\nabla \cdot \mathbf{j}_\perp = \nabla \cdot \mathbf{j} + \partial_t \rho = 0. \tag{2.8}$$

The name 'transverse' comes from the fact that in spatial Fourier components, the current $\mathbf{j}_\perp(\mathbf{k})$ must be perpendicular to $\mathbf{k}$. One also says that the vector potential in the Coulomb gauge is transverse because $\nabla \cdot \mathbf{A} = 0$. The transversality of the source term in (2.7) ensures that if $\mathbf{A}$ is transverse at one time, it is also transverse at all later times.

Conservation laws. Charge: $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$.

Energy (Poynting theorem):

$$\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} \tag{2.9}$$

$$u = \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \qquad \text{energy density} \tag{2.10}$$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \qquad \text{Poynting vector} \tag{2.11}$$

Momentum (see exercise).

2.1.2 Matter

Let us consider that all matter is made from charged point particles with charges e_α and positions $\mathbf{r}_\alpha$. The electric charge and current densities are then given by

$$\rho(\mathbf{x}, t) = \sum_\alpha e_\alpha \delta(\mathbf{x} - \mathbf{r}_\alpha(t)), \qquad \mathbf{j}(\mathbf{x}, t) = \sum_\alpha e_\alpha \mathbf{v}_\alpha(t) \delta(\mathbf{x} - \mathbf{r}_\alpha(t)) \tag{2.12}$$

The sum runs over all the particles. Charge conservation is ensured provided that $\dot{\mathbf{r}}_\alpha = \mathbf{v}_\alpha$.

The point charges are thus the 'sources' for the electromagnetic field. But their motion is also influenced by the fields via the Newton-Lorentz equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_\alpha \mathbf{v}_\alpha}{\sqrt{1-v_\alpha^2/c^2}} = e_\alpha\left(\mathbf{E}(\mathbf{r}_\alpha) + \mathbf{v}_\alpha \times \mathbf{B}(\mathbf{r}_\alpha)\right) \tag{2.13}$$

We use a relativistic framework here, and with the Lorentz factor, the time derivative is actually the one for the relativistic particle momentum. Note that in these equations, the coordinate $\mathbf{r}_\alpha$ enters generally in a nonlinear way. This differs from the Maxwell equations that are linear in the fields and potentials and whose solutions are linear in the charge and current distributions. (The superposition principle can be applied, see Eq.(2.6).) All nonlinear effects in optics can ultimately be traced back to the nonlinear response of matter to an applied electromagnetic field.

Quantization. The theory outlined so far describes physics at the end of the 19th century. It is unable to describe a stable state of matter because of the 'radiation catastrophe': positive and negative charges circle around each other, radiate electromagnetic waves and lose energy. Stable bound states exist only when the particle dynamics is quantized—recall the hydrogen atom.

Let us recall the typical energy and length scales for the hydrogen atom. The energy levels in Hydrogen are given by

$$E_n = -\frac{\mathrm{Ryd}}{n^2} = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2} \tag{2.14}$$

where the Rydberg constant is $1\,\mathrm{Ryd} \approx 13.6\,\mathrm{eV}$. (In cgs units, drop the factor $(4\pi\varepsilon_0)^2$.) The size of the Hydrogen atom is of the order of the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} \tag{2.15}$$

The typical wavelength of an electromagnetic wave resonant with a transition in Hydrogen is therefore of the order of

$$\lambda\!\!\!^{-} = \frac{\hbar c}{\mathrm{Ryd}} = \frac{2}{\alpha_{\mathrm{fs}}}a_0 \tag{2.16}$$

$$\frac{1}{\alpha_{\mathrm{fs}}} = \frac{4\pi\varepsilon_0\hbar c}{e^2} \approx 137 \tag{2.17}$$

Here, $\alpha_{\rm fs}$ is the fine structure constant. Its inverse can be understood as a measure of the speed of light in 'atomic units' (the natural units for the Hydrogen problem). The value $1/\alpha_{\rm fs} \approx 137$ is fairly large. This means two things:

— the size of the hydrogen atom is small compared to the wavelength of resonant radiation: the Lorentz force (2.13) has therefore only a weak dependence on $\mathbf{r}_\alpha$;

— the typical velocity of an electron in the Hydrogen atom is in the non-relativistic regime: we can therefore use non-relativistic mechanics to describe the matter response.

This will justify several approximations for the atom-light interaction that we are going to make in the rest of the lecture.

2.1.3 Lagrange-Hamilton formulation

Lagrangian. We now proceed to quantize the wave equation (2.7) in the 'canonical way'. The first step is to guess the corresponding Lagrangian. A good guess is the Lagrangian

$$L = -\sum_\alpha m_\alpha c^2 \sqrt{1 - \dot{\mathbf{r}}_\alpha^2/c^2} + \int \mathrm{d}^3x\, \mathcal{L}_{F+I} \tag{2.18}$$

where the Lagrangian density for the field and its interaction with the electric charges is given by

$$\mathcal{L}_{F+I} = \frac{\varepsilon_0}{2}\left(\dot{\mathbf{A}} + \nabla\phi\right)^2 - \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 - \phi\rho + \mathbf{A}\cdot\mathbf{j}. \tag{2.19}$$

Via the Euler-Lagrange equations, one gets the Maxwell equations (2.5,2.7) for the fields and the Newton-Lorentz equation (2.13) for the particles. Note that the Lagrangian (2.18,2.19) is invariant under gauge transformations

$$\phi \mapsto \phi - \partial_t\chi, \qquad \mathbf{A} \mapsto \mathbf{A} + \nabla\chi \tag{2.20}$$

where $\chi(\mathbf{x}, t)$ is an arbitrary smooth function. This gauge invariance is connected to charge conservation.

Coulomb gauge. We now proceed to specialize to the Coulomb gauge, simplify the Lagrangian and derive the Hamiltonian. We start with the terms involving the scalar potential in the Lagrangian (2.19). The mixed term is

$$\dot{\mathbf{A}} \cdot \nabla\phi = \nabla \cdot (\dot{\mathbf{A}}\phi) - \phi \nabla \cdot \dot{\mathbf{A}} \tag{2.21}$$

The first term is a divergence, and leads to a surface integral when integrated over the volume. We adopt the usual boundary condition that at the (infinitely remote) surface, the fields vanish: then this term is zero. The second term is zero in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$.

The term quadratic in the scalar potential is

$$(\nabla\phi)^2 = \nabla \cdot (\phi\nabla\phi) - \phi\triangle\phi = \nabla \cdot (\phi\nabla\phi) + \phi\rho/\varepsilon_0 \tag{2.22}$$

using the Laplace equation (2.5). The second term thus combines with the interaction part $-\phi\rho$ in the Lagrangian that becomes $-\frac{1}{2}\phi\rho$. This energy can be interpreted as the Coulomb interaction energy between the charges:

$$V_{\text{Coul}} = \frac{1}{2}\int \mathrm{d}^x\, \phi\rho = \frac{1}{2}\int \mathrm{d}^3x\, \mathrm{d}^3x'\, \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{4\pi\varepsilon_0\,|\mathbf{x}-\mathbf{x}'|} \tag{2.23}$$

$$= \frac{1}{2}\sum_{\alpha\beta} \frac{e_\alpha e_\beta}{4\pi\varepsilon_0\,|\mathbf{r}_\alpha - \mathbf{r}_\beta|} \tag{2.24}$$

where the factor $\frac{1}{2}$ ensures that all pairs of charges are only counted once. The divergent self-interaction for $\mathbf{r}_\alpha = \mathbf{r}_\beta$ also appears here. It is usually discarded. *The key point to note is that in the Coulomb gauge, the contribution of the scalar potential depends only on the particle coordinates. It is not a proper degree of freedom of the fields.*

To summarize, in the Coulomb gauge, the Lagrangian can be split into the following form:

$$L = -\sum_\alpha m_\alpha c^2\sqrt{1 - v_\alpha^2/c^2} - V_{\text{Coul}}(\{\mathbf{r}_\alpha\}) + \int \mathrm{d}^3x\, \mathcal{L}^\perp_{\text{F+I}} \tag{2.25}$$

with

$$\mathcal{L}^\perp_{\text{F+I}} = \frac{\varepsilon_0}{2}\dot{\mathbf{A}}^2 - \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 + \mathbf{A}\cdot\mathbf{j}. \tag{2.26}$$

We have added the subscript $_\perp$ to remind ourselves that this is only valid if the vector potential is transverse.

Hamiltonian. For the Hamiltonian, we need the canonical momenta conjugate to $\mathbf{r}_\alpha$ and $\mathbf{A}$:

$$\mathbf{p}_\alpha = \frac{\partial L}{\partial \dot{\mathbf{r}}_\alpha} = \frac{m\dot{\mathbf{r}}_\alpha}{\sqrt{1-\dot{\mathbf{r}}_\alpha^2/c^2}} + e_\alpha \mathbf{A}(\mathbf{r}_\alpha) \tag{2.27}$$

$$\mathbf{\Pi}(\mathbf{x}) = \frac{\delta L}{\delta \dot{\mathbf{A}}(\mathbf{x})} = \varepsilon_0 \dot{\mathbf{A}}(\mathbf{x}) \tag{2.28}$$

The particle momentum contains the relativistic kinetic momentum and an electromagnetic contribution. For the field momentum, it looks as if ε_0 where the 'mass' and $\dot{\mathbf{A}}$ the velocity.

Functional derivative. In the field case, we have a continuous collection of degrees of freedom, labelled by the space-points $\mathbf{x}$. This requires a generalization of the notion of a derivative to the infinite-dimensional case: the 'functional derivative' $\delta L/\delta \dot{\mathbf{A}}(\mathbf{x})$. It is the generalization of a gradient.

In mathematical terms, if we have a 'functional' $L[\mathbf{A}(\mathbf{x})]$, i.e. a mapping from the space of vector fields into the real numbers, its functional derivative is defined by the following approximation:

$$\begin{aligned} &L[\mathbf{A}(\mathbf{x}) + \delta\mathbf{A}(\mathbf{x})] \approx L[\mathbf{A}(\mathbf{x})] \\ &+ \int \mathrm{d}^3x \left.\frac{\delta L}{\delta \mathbf{A}(\mathbf{x})}\right|_{\mathbf{A}(\mathbf{x})} \delta\mathbf{A}(\mathbf{x}) + \mathcal{O}(\delta\mathbf{A}^2) \end{aligned} \tag{2.29}$$

Here, the second line is an example of a linear functional because the integral is linear in $\delta\mathbf{A}(\mathbf{x})$. The function with which the small deviation $\delta\mathbf{A}(\mathbf{x})$ is weighted under the integral *defines* the functional derivative. A mathematical theorem ensures that in a suitable space of functions, all linear functionals take this integral form.

It is a simple exercise to derive with (2.29) expression (2.28) for the field momentum $\mathbf{\Pi}$. We shall return to a less trivial example below.

The Hamiltonian is given by

$$H = \sum_\alpha \mathbf{p}_\alpha \cdot \dot{\mathbf{r}}_\alpha + \int \mathrm{d}^3x\, \mathbf{\Pi} \cdot \dot{\mathbf{A}} - L \tag{2.30}$$

where the field part is the obvious generalization to a continuous set of degrees of freedom. Putting everything together, we get

$$\begin{aligned} H &= \sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2c^4} + V_{\mathrm{Coul}}(\{\mathbf{r}_\alpha\}) \\ &\quad + \int \mathrm{d}^3x \left[\frac{\mathbf{\Pi}^2}{2\varepsilon_0} + \frac{(\nabla \times \mathbf{A})^2}{2\mu_0}\right] \end{aligned} \tag{2.31}$$

where the interaction between matter and (transverse) field arises due to the 'minimal coupling' prescription only. (The term linear in $\mathbf{j}$ in the Lagrangian cancels with the term linear in $\dot{\mathbf{r}}_\alpha$ in (2.30).)

Canonical equations. The motion of particles and fields in the Hamiltonian formalism can be described in a compact way in terms of Poisson brackets. We discuss this in some detail because they provide another example of functional derivatives and because they bear strong similarities to the commutators of the quantum theory. In addition, it turns out to be tricky to get the transverse wave equation (2.7).

The Poisson bracket provides the time evolution of any function Q (or functional) of the coordinates and momenta by

$$\dot{Q} = \{H,\, Q\} \tag{2.32}$$

where we define

$$\begin{aligned} \{A,\, B\} \;&=\; \sum_\alpha \frac{\partial A}{\partial \mathbf{p}_\alpha}\cdot\frac{\partial B}{\partial \mathbf{r}_\alpha} - \frac{\partial A}{\partial \mathbf{r}_\alpha}\cdot\frac{\partial B}{\partial \mathbf{p}_\alpha} \\ &\quad + \int \mathrm{d}^3x \left[\frac{\delta A}{\delta \mathbf{\Pi}(\mathbf{x})}\cdot\frac{\delta B}{\delta \mathbf{A}(\mathbf{x})} - \frac{\delta A}{\delta \mathbf{A}(\mathbf{x})}\cdot\frac{\delta B}{\delta \mathbf{\Pi}(\mathbf{x})}\right] \end{aligned} \tag{2.33}$$

Here, functional derivatives with respect to $\mathbf{A}$ and $\mathbf{\Pi}$ appear for the fields. The Poisson bracket is antisymmetric in A and B and satisfies a 'Jacobi identity' (as does the commutator).

By working out $\{H, \mathbf{r}\}$, only the second term of the first line in (2.33) contributes, and we get (after some calculations) the relativistic relation between velocity and momentum, Eq.(2.27). This is left as an exercise. Similarly, one gets $\dot{\mathbf{A}} = \mathbf{\Pi}/\varepsilon_0$.

A more complicated calculation is needed for $\dot{\mathbf{\Pi}}$ where we have to evaluate

$$-\frac{\delta}{\delta \mathbf{A}(\mathbf{x})}\sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4} \tag{2.34}$$

We note first that

$$\frac{\delta \mathbf{A}(\mathbf{r}_\alpha)}{\delta \mathbf{A}(\mathbf{x})} = \delta(\mathbf{r}_\alpha - \mathbf{x}) \tag{2.35}$$

because the 'evaluation functional' $\mathbf{A} \mapsto \mathbf{A}(\mathbf{r}_\alpha)$ is of course a linear functional. (For mathematicians, this property *defines* the δ-function.) To proceed, we use

the usual rules of differential calculus and get

$$-\frac{\delta}{\delta\mathbf{A}(\mathbf{x})}\sqrt{[\mathbf{p}_\alpha - e_\alpha\mathbf{A}(\mathbf{r}_\alpha)]^2c^2 + m^2c^4} = \delta(\mathbf{r}_\alpha - \mathbf{x})\frac{e_\alpha(\mathbf{p}_\alpha - e_\alpha\mathbf{A}(\mathbf{r}_\alpha))c^2}{\sqrt{[\mathbf{p}_\alpha - e_\alpha\mathbf{A}(\mathbf{r}_\alpha)]^2c^2 + m^2c^4}} \tag{2.36}$$

Performing the same calculations as for the particles' equations of motion, this can be written as the current density $\mathbf{j}(\mathbf{x})$.

The last term now involves the derivative

$$-\frac{\delta}{\delta\mathbf{A}(\mathbf{x})}\frac{1}{2\mu_0}\int \mathrm{d}^3x(\nabla\times\mathbf{A})^2 \tag{2.37}$$

that we handle with the mathematical definition (2.29). Consider a small change $\mathbf{a}(\mathbf{x})$ of the vector potential. To linear order, this gives a change

$$\begin{aligned}&(\nabla\times(\mathbf{A}+\mathbf{a}))^2 - (\nabla\times\mathbf{A})^2 \approx 2(\nabla\times\mathbf{a})\cdot(\nabla\times\mathbf{A})\\ &= 2\nabla\cdot[\mathbf{a}\times(\nabla\times\mathbf{A})] + 2\mathbf{a}\cdot[\nabla\times(\nabla\times\mathbf{A})]\end{aligned} \tag{2.38}$$

the first term is a divergence and vanishes after integrating over all space. The second one contains $\mathbf{a}$ multiplied with a weighting function so that we get

$$-\frac{1}{2\mu_0}\frac{\delta}{\delta\mathbf{A}(\mathbf{x})}\int \mathrm{d}^3x(\nabla\times\mathbf{A})^2 = -\frac{1}{\mu_0}\nabla\times(\nabla\times\mathbf{A}) \tag{2.39}$$

Putting everything together, we have for the equation of motion of the vector potential:

$$\dot{\mathbf{\Pi}} = \varepsilon_0\ddot{\mathbf{A}} = \mathbf{j}(\mathbf{x}) - \frac{1}{\mu_0}\nabla\times(\nabla\times\mathbf{A}) \qquad \ldots\text{wrong} \tag{2.40}$$

which is nearly equivalent to the wave equation (2.7). The point is that the source term is the 'full current', not its transverse part. This is actually an error in our calculation because we did not take into account the fact that the vector potential is restricted to be transverse.

A simple way to repair this is to use $\mathbf{A}_\perp$ in the Lagrangian (2.25) and the Hamiltonian (2.31). Now, the link between $\mathbf{A}_\perp$ and the 'full' $\mathbf{A}$ is a linear functional (actually, a linear projector). This can be seen in the following way: consider an arbitrary $\mathbf{A}$ and perform a gauge transformation (2.20) to remove the nonzero divergence. This fixes the 'gauge function' to satisfy

$$\triangle\chi = -\nabla\cdot\mathbf{A} \tag{2.41}$$

whose solution (vanishing at infinity) is given by a 'superposition of Coulomb potentials':

$$\chi(\mathbf{x}) = \frac{1}{4\pi}\int \mathrm{d}^3x' \, \frac{\nabla' \cdot \mathbf{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{2.42}$$

where ∇' means the gradient with respect to $\mathbf{x}'$. After the gauge transformation, the now transverse vector potential is given by

$$\mathbf{A}_\perp(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \frac{1}{4\pi}\int \mathrm{d}^3x' \, [\nabla' \cdot \mathbf{A}(\mathbf{x}')]\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{2.43}$$

Now, $\nabla(1/r) = -\nabla'(1/r)$, and after one integration by parts we have

$$A_{\perp i}(\mathbf{x}) = \int \mathrm{d}^3x' \, \delta^\perp_{ij}(\mathbf{x}-\mathbf{x}')A_j(\mathbf{x}) \tag{2.44}$$

where the so-called 'transverse δ-function' is given by

$$\delta^\perp_{ij}(\mathbf{x}-\mathbf{x}') = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{4\pi}\partial'_j\partial'_i \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{2.45}$$

A careful evaluation of the second derivatives yields the explicit result

$$\delta^\perp_{ij}(\mathbf{x}-\mathbf{x}') = \frac{2}{3}\delta_{ij}\delta(\mathbf{x}-\mathbf{x}') - \frac{1}{4\pi}\left(\frac{\delta_{ij}}{r^3} - 3\frac{r_i r_j}{r^5}\right), \qquad \mathbf{r} = \mathbf{x}-\mathbf{x}' \tag{2.46}$$

By construction, the transverse δ-function acts like a usual δ-function on vector fields that are already transverse. We can interpret it as the 'unit operator' in the space of transverse vector fields.

Finally, if we write $\mathbf{A}_\perp$ in the Hamiltonian, the equations of motion for $A_{\perp i}(\mathbf{x})$ lead to the following term

$$\int \mathrm{d}^3x' \frac{\delta H}{\delta \Pi_j(\mathbf{x}')}\frac{\delta A_{\perp i}(\mathbf{x})}{\delta A_j(\mathbf{x}')} = \int \mathrm{d}^3x' \frac{\delta H}{\delta \Pi_j(\mathbf{x}')}\delta^\perp_{ij}(\mathbf{x}-\mathbf{x}') = \left(\frac{\delta H}{\delta \mathbf{\Pi}(\mathbf{x}')}\right)_{\perp i} \tag{2.47}$$

The functional derivative with respect to $\mathbf{A}$ makes the transverse δ-function appear. The integral over $\mathbf{x}'$ then projects the functional derivative with respect to $\mathbf{\Pi}$ into the transverse subspace. In this way, the time derivative of $\mathbf{A}_\perp$ is transverse, as it should do in order to maintain the Coulomb gauge at all times.

The same procedure applies to the equation of motion for the field momentum: the bottomline is that the source current $\mathbf{j}$ is 'transversalized', leading to the correct wave equation (2.7).

2.1.4 Quantization

Mode expansion. The next step is to look for 'normal modes' of this field theory. We first identify some general requirements for the modes. The passage to plane waves is a little bit tricky and is treated in detail in Sec. 2.1.5. Note that we ignore for the moment the matter-field coupling: wo focus on the field Hamiltonian only.

We adopt the expansion

$$\begin{pmatrix} \mathbf{A}(\mathbf{x},t) \\ \mathbf{\Pi}(\mathbf{x},t) \end{pmatrix} = \sum_\kappa \mathbf{f}_\kappa(\mathbf{x}) \begin{pmatrix} q_\kappa(t) \\ p_\kappa(t) \end{pmatrix} \tag{2.48}$$

where the 'mode functions' $\mathbf{f}_\kappa(\mathbf{x})$ carry the space dependence and the 'coordinates' $q_\kappa(t)$ and 'momenta' $p_\kappa(t)$ the time-dependence.

The Coulomb gauge requires, of course, $\nabla \cdot \mathbf{f}_\kappa(\mathbf{x}) = 0$: the mode functions must be transverse.

We want the Hamiltonian to adopt a simple form using these modes. The space integral over $\mathbf{\Pi}^2$ becomes simple if we impose the modes to be orthogonal:

$$\int \mathrm{d}^3x\, \mathbf{f}_\kappa(\mathbf{x}) \cdot \mathbf{f}_{\kappa'}(\mathbf{x}) = N_\kappa \delta_{\kappa\kappa'} \tag{2.49}$$

where N_κ is a normalization constant that we fix later. The momentum-part of the Hamiltonian then becomes

$$\frac{1}{2\varepsilon_0} \int \mathrm{d}^3x\, \mathbf{\Pi}^2 = \sum_\kappa N_\kappa \frac{p_\kappa^2}{2\varepsilon_0} \tag{2.50}$$

Similarly, for the integral over $(\nabla \times \mathbf{A})^2$. Integrating by parts:

$$\begin{aligned} &\int \mathrm{d}^3x\, (\nabla \times \mathbf{f}_\kappa) \cdot (\nabla \times \mathbf{f}_{\kappa'}) \\ &= \int \mathbf{dA} \cdot [\mathbf{f}_\kappa \times (\nabla \times \mathbf{f}_{\kappa'})] + \int \mathrm{d}^3x\, \mathbf{f}_\kappa \cdot [\nabla \times (\nabla \times \mathbf{f}_{\kappa'})] \end{aligned} \tag{2.51}$$

The volume integral, strictly speaking, must be evaluated over a finite volume only, otherwise, we could not work with a discrete set of mode labels κ. The boundary term can nevertheless be made to vanish if either (i) the mode function $\mathbf{f}_\kappa$ or its curl $(\nabla \times \mathbf{f}_{\kappa'})$ is required to vanish on the boundary of the volume or (ii) periodic boundary conditions on 'opposite faces' of a cubic volume are assumed. The case (i) is appropriate for modes in a cavity with perfectly conducting boundaries: then, $\mathbf{f}_\kappa$ is proportional to the electric field, and the integrand in

Eq.(2.51) vanishes because the field is normal to the boundary. The case (ii) is the favorite one for theorists because the mode functions can be taken as plane waves. Note that the eigenfrequencies of the two cavities are not the same. We ignore for the moment the complications of complex mode functions (see details below and the exercises) and continue.

The volume integral in (2.51) is reduced to the orthogonality relation if we require the mode functions to be eigenfunctions of the (vector) Helmholtz equation:

$$\nabla \times (\nabla \times \mathbf{f}_\kappa) = \varepsilon_0 \mu_0 \omega_\kappa^2 \mathbf{f}_\kappa \tag{2.52}$$

(This equation is actually equivalent to the scalar Helmholtz equation for all components of $\mathbf{f}_\kappa$ because of transversality.)

With all these assumptions taken together, the Hamiltonian for the field takes the form

$$H_{\mathrm{F}} = \sum_\kappa \left[\frac{N_\kappa}{2\varepsilon_0} p_\kappa^2 + \frac{\varepsilon_0}{2} N_\kappa \omega_\kappa^2 q_\kappa^2 \right] \tag{2.53}$$

We now fix the normalization to be $N_\kappa = 1$ and get a sum of harmonic oscillator Hamiltonians, one for each mode κ with 'mass' ε_0 and frequency ω_κ.

Note: the construction of field modes is a 'classical problem' of electrodynamics, it has nothing to do with quantum mechanics. The word 'quantization volume' that is sometimes used (to ensure that the mode index κ is discrete) is therefore misleading. Quantization is something different, as we shall see now.

Mode operators. Quantization proceeds by promoting the p_κ and q_κ to operators with the commutation relations

$$\frac{\mathrm{i}}{\hbar} [p_\kappa,\, q_{\kappa'}] = \delta_{\kappa\kappa'} \tag{2.54}$$

The choice for this commutator is similar to the one for the particle coordinates and momenta in ordinary quantum mechanics. So in the end, field quantization is nothing else but ordinary quantization, once the dynamics of the field is reduced to a discrete set of 'normal modes'. The procedure that we have followed was first laid out by Dirac. It is called 'canonical quantization'.

The commutator between the fields becomes

$$\frac{\mathrm{i}}{\hbar} [\Pi_i(\mathbf{x}),\, A_j(\mathbf{x}')] = \sum_\kappa f_{\kappa i}(\mathbf{x}) f_{\kappa j}(\mathbf{x}') \stackrel{!}{=} \delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') \tag{2.55}$$

The last equality is obtained by applying the canonical quantization scheme directly to the fields, care being taken that the fields (operators) 'live' in the space

of transverse vector fields (operators). It means that the mode functions $\mathbf{f}(\mathbf{x})$ form a *complete set* of functions in the transverse field space. To implement this equality, one takes in practice the limit of an infinitely large quantization volume where the sum degenerates into an integral. With discrete functions, one can actually represent only a 'finite volume version' of the transverse δ-function—the one that comes by applying the finite volume boundary conditions to the equation (2.41).

For the harmonic oscillator, creation and annihilation operators are a convenient tool to construct the Hilbert space of quantum states. In our context, these operators, $a_\kappa^\dagger$ and a_κ, correspond to the 'creation' and 'destruction' of one 'photon'. The mode coordinate and momentum operators are given by

$$q_\kappa = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa}}\left(a_\kappa + a_\kappa^\dagger\right) \tag{2.56}$$

$$p_\kappa = \sqrt{\frac{\hbar\varepsilon_0\omega_\kappa}{2}}\mathrm{i}\left(a_\kappa^\dagger - a_\kappa\right) \tag{2.57}$$

where we continue to write ε_0 for the oscillator mass and where the commutation relation is

$$\left[a_\kappa,\, a_{\kappa'}^\dagger\right] = \delta_{\kappa\kappa'} \tag{2.58}$$

The field Hamiltonian then takes the form

$$H_\mathrm{F} = \sum_\kappa \frac{\hbar\omega_\kappa}{2}\left(a_\kappa a_\kappa^\dagger + a_\kappa^\dagger a_\kappa\right) = \sum_\kappa \hbar\omega_\kappa\left(a_\kappa^\dagger a_\kappa + \tfrac{1}{2}\right) \tag{2.59}$$

The last way of writing makes two essential things explicit:

— the energies (the energy eigenvalues!) of a given field mode are quantized in units of $\hbar\omega_\kappa$ (a 'photon energy') and the 'number of photons' is represented by the operator $a_\kappa^\dagger a_\kappa$.

— The ground state of the field corresponds to the state $|\mathrm{vac}\rangle$ such that $a_\kappa|\mathrm{vac}\rangle = 0$ for all κ. This is an energy eigenstate whose energy is infinite, $\frac{1}{2}\sum_\kappa \hbar\omega_\kappa$, the sum over the 'zero-point energies' of all the modes.

To summarize, we give the mode expansions of the vector potential that we have found

$$\mathbf{A}(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa}}\mathbf{f}_\kappa(\mathbf{x})\left(a_\kappa \mathrm{e}^{-\mathrm{i}\omega_\kappa t} + a_\kappa^\dagger \mathrm{e}^{\mathrm{i}\omega_\kappa t}\right). \tag{2.60}$$

We have used here the Heisenberg picture for the vector potential operator. From the Hamiltonian (2.59), it is easy to show that the operator $a_\kappa(t)$ evolves with a complex exponential $\mathrm{e}^{-\mathrm{i}\omega_\kappa t}$. This is also called a 'positive frequency operator'. It can be shown that operators that destroy particles and lower the energy of a quantum state are always positive frequency operators.

Plane wave modes. For completeness, we give here the plane-wave expansion for the field mode functions. These are complex, and therefore they are normalized according to

$$\int \mathrm{d}^3x\, \mathbf{f}^*_\kappa(\mathbf{x}) \cdot \mathbf{f}_{\kappa'}(\mathbf{x}) = \delta_{\kappa\kappa'} \tag{2.61}$$

instead of Eq.(2.49). This can be ensured with the choice

$$\mathbf{f}_\kappa(\mathbf{x}) = \frac{1}{\sqrt{V}} \mathbf{u}_\kappa \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.62}$$

where V is the volume of the box with periodic boundary conditions, $\mathbf{k}$ is a discrete wave vector: it increases in steps of $2\pi/V^{1/3}$ for a cubic box. And $\mathbf{u}_\kappa$ is a 'transverse' polarization vector with the property $\mathbf{k}\cdot\mathbf{u}_\kappa = 0$. There are two mutually orthogonal choices of polarization for a given $\mathbf{k}$. (These can be complex, describing circular polarization.) The frequency of this mode is $\omega_\kappa = |\mathbf{k}|/\sqrt{\varepsilon_0\mu_0} = c|\mathbf{k}|$, as can be seen from the Helmholtz equation (2.52). The magnetic field is oriented along $\mathbf{k}\times\mathbf{u}_\kappa \equiv (\omega_\kappa/c)\mathbf{v}_\kappa$.

Finally, the quantized vector potential, electric and magnetic fields are given in terms of the following plane wave expansion

$$\mathbf{A}_\perp(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa V}} \left(\mathbf{u}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_\kappa t)} a_\kappa + \text{h.c.}\right). \tag{2.63}$$

$$\mathbf{E}_\perp(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar\omega_\kappa}{2\varepsilon_0 V}} \left(\mathrm{i}\mathbf{u}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_\kappa t)} a_\kappa + \text{h.c.}\right). \tag{2.64}$$

$$\mathbf{B}(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar\omega_\kappa\mu_0}{2V}} \left(\mathrm{i}\mathbf{v}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_\kappa t)} a_\kappa + \text{h.c.}\right). \tag{2.65}$$

Sometimes, you may encounter these formulas without the factor i. Then the operators $\mathrm{i}a_{\mathbf{k}\lambda}$ are being used instead, but they have the same commutation relations. Note that Eq.(2.64) gives only the 'transverse' part of the electric field. The 'longitudinal' part, $-\nabla\phi$, is determined according to (2.5) by the charge density.

Note: A useful shortcut to derive the prefactors is the following: for each mode, match the energy density $\frac{1}{2}\varepsilon_0\mathbf{E}^2 + (1/2\mu_0)\mathbf{B}^2$ to the photon energy per

quantization volume, $(\hbar\omega_\kappa/V)(a_\kappa^\dagger a_\kappa + \frac{1}{2})$. In free space, the electric and magnetic energy densities are equal. Average over the spatial oscillations in $\mathbf{E}^2$ for simplicity.

Exercise. Show that the momentum of the field can be written as a sum over modes as well. With plane wave modes, the momentum per mode is quantized in units of $\hbar\mathbf{k}$, as expected. Arbitrary cavity modes involving sin or cos functions are not eigenfunctions of the momentum operators, therefore their momentum is not well-defined. A 'cavity photon' therefore does not have a well-defined momentum.

Exercise. Write the equation of motion for a mode operator a_κ and include the source current. Solve it for known time-dependence of current.

2.1.5 Plane wave quantization

In many cases, plane waves are a more natural choice for the mode functions. But they are complex, and this leads to some technical difficulties. We outline below a quantization procedure that directly starts from plane waves.

Plane waves are an obvious choice in a box of volume V with periodic boundary conditions. We can expand

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t)\,\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \tag{2.66}$$

where the plane waves are normalized to unity in the ('quantization') volume V. Some simple manipulations then lead to the Lagrangian

$$L = \frac{\varepsilon_0}{2}\sum_{\mathbf{k}}\left(\dot{\mathbf{A}}_{\mathbf{k}}\cdot\dot{\mathbf{A}}_{-\mathbf{k}} - c^2k^2\,\mathbf{A}_{\mathbf{k}}\cdot\mathbf{A}_{-\mathbf{k}} + \mathbf{A}_{\mathbf{k}}\cdot\mathbf{j}_{-\mathbf{k}}\right). \tag{2.67}$$

This already decomposes into a sum over the wave vectors $\mathbf{k}$. Only waves with $\mathbf{k}$ and $-\mathbf{k}$ are coupled. We can also decompose the complex vector in a basis of orthogonal polarization vectors (there are two of them with $\mathbf{k}\perp\mathbf{u}_{\mathbf{k}\lambda}$)

$$\mathbf{A}_{\mathbf{k}} = \sum_\lambda \mathbf{u}_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda}. \tag{2.68}$$

For simplicity, we assume that the polarization vectors are real and $\mathbf{u}_{\mathbf{k}\lambda} = \mathbf{u}_{-\mathbf{k}\lambda}$. The complex number $A_{\mathbf{k}\lambda}$ still encodes two degrees of freedom that are related to the plane waves with wave vectors $\mathbf{k}$ and $-\mathbf{k}$. This is apparent from the relation

$$\mathbf{A}_{\mathbf{k}}^* = \mathbf{A}_{-\mathbf{k}} \tag{2.69}$$

that follows from the fact that $\mathbf{A}(\mathbf{x},t)$ is real. We now make the decomposition

$$A_{\mathbf{k}\lambda} = \frac{q_1 + \mathrm{i}q_2}{\sqrt{2}}, \quad A_{-\mathbf{k}\lambda} = \frac{q_1 - \mathrm{i}q_2}{\sqrt{2}}, \tag{2.70}$$

where the indices $\mathbf{k}\lambda$ have been dropped just for the clarity of presentation. In terms of these variables, the terms with $\mathbf{k}$ and $-\mathbf{k}$ combine to give a sum of two harmonic oscillators:

$$L_{\mathbf{k}} = \frac{\varepsilon_0}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 - c^2k^2\left(q_1^2 + q_2^2\right) + \text{interactions}\right) \tag{2.71}$$

(Note that both $\mathbf{k}$ and $-\mathbf{k}$ give the same contribution.)

Quantization. Canonical quantization means that the coordinate q_1 and its canonically conjugate momentum

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \varepsilon_0 \dot{q}_1 \tag{2.72}$$

are replaced by operators with the commutator $[p_\alpha,\, q_\beta] = -\mathrm{i}\hbar\,\delta_{\alpha\beta}$ $(\alpha,\beta = 1,2)$. The Hamiltonian is given by

$$H = \frac{1}{2\varepsilon_0}\left(p_1^2 + p_2^2\right) + \frac{\varepsilon_0}{2}c^2k^2\left(q_1^2 + q_2^2\right) + \text{interaction} \tag{2.73}$$

so that we are dealing with two uncoupled harmonic oscillators. We can use the well-known results from the Quantum Mechanics lecture if we write $\omega = ck$ and $m = \varepsilon_0$. Here ω is really a frequency (the light frequency for a plane wave), and m is not a mass because q is not a position coordinate, but has the dimensions of a vector potential.

The quantization of the harmonic oscillator is most easily written down in terms of the annhilation and creation operators a_α and $a_\alpha^\dagger$. These are dimensionless quantities with commutator $\left[a_\alpha,\, a_\beta^\dagger\right] = \delta_{\alpha\beta}$. They allow to represent coordinates and momenta as

$$q_\alpha = \sqrt{\frac{\hbar}{2\varepsilon_0\omega}}\left(a_\alpha + a_\alpha^\dagger\right), \tag{2.74}$$

$$p_\alpha = -\mathrm{i}\sqrt{\frac{\hbar\varepsilon_0\omega}{2}}\left(a_\alpha - a_\alpha^\dagger\right). \tag{2.75}$$

We thus get the following expansion of the vector potential

$$\begin{aligned}\mathbf{A}(\mathbf{x}) = {}' \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar}{4\varepsilon_0 V\omega_{\mathbf{k}}}}\mathbf{u}_{\mathbf{k}\lambda}\Big\{&\left(a_1 + a_1^\dagger + \mathrm{i}\left(a_2 + a_2^\dagger\right)\right)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \\ &+ \left(a_1 + a_1^\dagger - \mathrm{i}\left(a_2 + a_2^\dagger\right)\right)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\Big\}\end{aligned} \tag{2.76}$$

The primed sum is to remind us that we sum only over 'one half' of k-space, regrouping $\mathbf{k}$ and $-\mathbf{k}$. We now introduce the operators

$$b_1 = \frac{a_1 + \mathrm{i}a_2}{\sqrt{2}}, \quad b_2 = \frac{a_1 - \mathrm{i}a_2}{\sqrt{2}}. \tag{2.77}$$

They also satisfy the commutation relations $\left[b_\alpha,\, b_\beta^\dagger\right] = \delta_{\alpha\beta}$ and are therefore equally well suited to represent the normal modes of the field. They have the advantage that the plane wave expansion (2.76) can be written in the form

$$\mathbf{A}(\mathbf{x}) \;=\; \sum_{\mathbf{k},\lambda}{}' \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\mathbf{k}}}}\mathbf{u}_{\mathbf{k}\lambda}\left\{b_1 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + b_2 \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.}\right\}. \tag{2.78}$$

We can now label the mode operators as $b_1 = a_{\mathbf{k}\lambda}$ and $b_2 = a_{-\mathbf{k}\lambda}$, so that the summation can be extended over all k-space:

$$\boxed{\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\mathbf{k}}}}\left\{\mathbf{u}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.}\right\}.} \tag{2.79}$$

This is the standard expansion of the vector potential in quantum electrodynamics.

Interaction and free field Hamiltonian. The interaction Hamiltonian with the transverse current takes the following form in the quantized theory:

$$H_{FA} \;=\; \int \mathrm{d}^3x\, \mathbf{A}(\mathbf{x}) \cdot \mathbf{j}_\perp(\mathbf{x}, t) \tag{2.80}$$

$$\;=\; \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_{\mathbf{k}}}}\left\{a_{\mathbf{k}\lambda}\mathbf{u}_{\mathbf{k}\lambda} \cdot \mathbf{j}^*_{\perp,\mathbf{k}}(t) + \mathrm{h.c.}\right\}, \tag{2.81}$$

where $\mathbf{j}_{\perp,\mathbf{k}}(t)$ is the spatial Fourier transform.

The dynamics of the field itself is generated by the well-known collection of harmonic oscillators:

$$H_F = \sum_{\mathbf{k},\lambda} \hbar\omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \tfrac{1}{2}\right). \tag{2.82}$$

In the present context, we treated the current density as an 'external', given function with an explicit time-dependence. For this reason, there is no Hamiltonian required for it. If the motion of the charges is quantized as well, one needs, of course, an 'atomic' Hamiltonian to generate their dynamics.

Field commutators. From the mode expansions (2.79, 2.64, 2.65) and the commutation relations between the annihilation operators, one can compute the commutator between the electric and magnetic fields. In the Lagrangian description, one finds already that the vector potential and the electric field are (up to a factor $-\varepsilon_0$) canonically conjugate variables. This is only true in the space of 'transverse vector functions' (fields with zero divergence), however. In the quantized theory, the corresponding commutator is

$$[A_l(\mathbf{x},t),\, E_m(\mathbf{x}',t)] = \frac{\mathrm{i}\hbar}{\varepsilon_0}\delta^{\perp}_{lm}(\mathbf{x}-\mathbf{x}') \tag{2.83}$$

where $\delta^{\perp}_{lm}$ is the 'transverse δ-function' defined in Eq.(2.45). As discussed above, this distribution acts like a δ-function on fields with zero divergence, and projects an arbitrary vector field $\mathbf{F}$ on its transverse part $\mathbf{F}^{\perp}$

$$F^{\perp}_l(\mathbf{x}) = \int \mathrm{d}^3x'\, \delta^{\perp}_{lm}(\mathbf{x}-\mathbf{x}')\, F_m(\mathbf{x}') \tag{2.84}$$

By definition, the transverse part has zero divergence, $\nabla\cdot\mathbf{F}^{\perp} = 0$. The projection is most easily constructed in Fourier space

$$F^{\perp}_j(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \frac{k^2\delta_{jl} - k_jk_l}{k^2}\tilde{F}_l(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.85}$$

where $\tilde{\mathbf{F}}(\mathbf{k})$ is the spatial Fourier transform of $\mathbf{F}(\mathbf{x})$. This relation allows to deduce explicit expressions for the transverse δ-function.

Note that in the wave equation (2.7) for the vector potential, we have an alternative relation between the current density $\mathbf{j}$ and its transverse part:

$$\mathbf{j}_{\perp} = \mathbf{j} - \varepsilon_0\partial_t\nabla\phi,$$

The 'longitudinal part' that is subtracted here is thus related to the electrostatic field created by the corresponding charge density. More details are discussed in the exercises.

2.2 Photons and the quantum vacuum

2.2.1 'Photons'

The present quantized description of the electromagnetic field allows us to give a more precise meaning to the word 'photon'. *A photon is an excitation of a mode*

of the field. We have seen that the quantized field reduces to a collection of harmonic oscillators, one for each mode. As we know from the harmonic oscillator, its stationary states are labelled by non-negative numbers $n = 0, 1, \ldots$. One says that in these states, the mode contains 'n photons'. The creation operator $a^\dagger$ whose action on these states is $a^\dagger|n\rangle \propto |n+1\rangle$, thus 'creates one photon'. This picture is consistent with the assumption (dating back to Einstein (1905)) that photons correspond to 'energy packets' of $\hbar\omega_k$ of the electromagnetic field. If a plane wave mode expansion is used, we can also say that the momentum of a photon is $\hbar\mathbf{k}$, as we know from de Broglie (1926) or from the Compton effect.

It is however possible to use different mode expansions for the same field. For example, we could have used an expansion in terms of spherical vector harmonics which differs from the plane wave expansion by a (infinite-dimensional) unitary transformation. A single-photon state in the plane wave basis thus becomes a superposition of single-photon states in infinitely many spherical modes. Conversely, a 'photon' in this description would not correspond to a plane wave (its momentum would not be $\hbar\mathbf{k}$), but it would have a definite angular momentum with respect to the origin. It is even possible to define photons that are wavepackets localized in time, by superposing plane waves with neighboring frequencies. This picture allows to describe experiments with 'single-photon pulses'. We refer to the exercises to look at these properties in more detail.

Finally, we precall that the computation of a suitable set of mode functions is a 'classical' problem: no quantum theory is needed to state it. Using a cubic 'quantization' box, one can show that the plane waves with wave vectors $\mathbf{k} = (2\pi/L)(n_x, n_y, n_z)^T$ ($n_i \in \mathbb{Z}$) are orthogonal with respect to the scalar product

$$\int_{L^3} \mathrm{d}^3x\, \mathbf{F}^*(\mathbf{x}) \cdot \mathbf{G}(\mathbf{x}). \tag{2.86}$$

This is clear for different wave vectors, $\mathbf{k} \neq \mathbf{k}'$. But for a given $\mathbf{k}$, one can also find two orthogonal polarization vectors $\boldsymbol{\varepsilon}_{1,2}$ that give orthogonal modes (the scalar product $\boldsymbol{\varepsilon}_1^* \cdot \boldsymbol{\varepsilon}_2$ is zero). The corresponding magnetic field mode functions are also orthogonal. The electrodynamics becomes a 'quantum' theory only when the amplitudes of the mode functions become suitably normalized operators.

2.2.2 The Fock-Hilbert space

The Hilbert space of the quantized field is constructed from the mode operators a_κ and $a_\kappa^\dagger$. The state of lowest energy is called the 'vacuum state', $|\mathrm{vac}\rangle$ or $|0\rangle$:

$$a_\kappa|\mathrm{vac}\rangle = 0 \tag{2.87}$$

The 'one-photon sector' is spanned by the infinitely many states with one photon per mode

$$|1_\kappa\rangle = a_\kappa^\dagger |\mathrm{vac}\rangle \tag{2.88}$$

...and so on. A typical state can be labelled by its 'occupation numbers' $|n_1, \ldots n_\kappa, \ldots\rangle = |\{n_\kappa\}\rangle$, it contains n_κ photons in the mode κ. The so-called Fock-Hilbert space is generated by taking linear combinations of these basis vectors with complex coefficients. From linear combinatinons with a finite number of terms, one completes the space by limiting procedures with respect to a suitable topology (related to the usual scalar product). Since the field theory ultimately contains a infinitely many modes and even a continuum, the topological structure can be quite intricate.

The field operators (vector potential, electric and magnetic fields) act between the N- and $N \pm 1$-photon sectors via the ladder operators a_κ and $a_\kappa^\dagger$ they contain. Their expectation values in the state $|\{n_\kappa\}\rangle$ are thus zero. To get a nonzero expectation value, one must construct superpositions of number states with different particle numbers. These states are not stationary in general. In quantum optics, an example of such states are the coherent states, useful to describe classical fields or to approximate a laser field. In high-energy physics, one usually discards such superpositions by a 'super-selection rule': one argues that for massive particles, the rest mass is so large that the relative phase between the components that differ in particle number varies so rapidly that one cannot distinguish, in practice, between a superposition state and a mixed state (to be described by a density matrix, see below).

2.2.3 Vacuum fluctuations

In the vacuum state, the expectation value $\langle \mathbf{E}(\mathbf{x}, t)\rangle = \langle \mathrm{vac}|\mathbf{E}(\mathbf{x}, t)|\mathrm{vac}\rangle = 0$ because a_κ annihilates the vacuum state and $a_\kappa^\dagger$ can be made to act to the left on the vacuum state which is annihilated. **Note.** The same is true for any stationary (or number) state, see Sec. 2.2.2 below.

The vacuum fluctuations become visible in the next moment of the field: for a given mode κ,

$$\frac{\hbar\omega_\kappa}{2\varepsilon_0}\langle(\mathbf{f}_\kappa(\mathbf{x})a(t) + \text{h.c.})^2\rangle = \frac{\hbar\omega_\kappa}{2\varepsilon_0}|\mathbf{f}_\kappa(\mathbf{x})|^2\langle a_\kappa(t)a_\kappa^\dagger(t)\rangle = \frac{\hbar\omega_\kappa}{2\varepsilon_0}|\mathbf{f}_\kappa(\mathbf{x})|^2 \tag{2.89}$$

in the last step, we have used that $a_\kappa(t) = a_\kappa(0)\,\mathrm{e}^{-\mathrm{i}\omega_\kappa t}$. In the case of plane wave modes, we have $|\mathbf{f}_\kappa(\mathbf{x})|^2 = 1/V$. The sum over wave vectors $\mathbf{k}$ and polarization

indices λ can be written in the form

$$\langle \mathbf{E}^2(\mathbf{x},t)\rangle = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\frac{\hbar\omega_\kappa}{2\varepsilon_0}\rho(\omega) \tag{2.90}$$

where $\rho(\omega)$ is the so-called 'local density of modes' (per unit frequency and unit volume):

$$\rho(\mathbf{x};\omega) = 2\pi\sum_\kappa |\mathbf{f}_\kappa(\mathbf{x})|^2\delta(\omega-\omega_\kappa) = \frac{2\pi}{V}\sum_\kappa \delta(\omega-\omega_\kappa) \tag{2.91}$$

For the plane wave modes, ω_κ only depends on the magnitude of $\mathbf{k}$, and in the continuum limit,

$$\sum_{\mathbf{k}} = V\int\frac{\mathrm{d}^3k}{(2\pi)^3} \tag{2.92}$$

we get after a simple integration

$$\rho(\omega) = \frac{2}{\pi}\frac{\omega^2}{c^3} \tag{2.93}$$

The full vacuum fluctuation of the electric field is then infinite

$$\langle \mathbf{E}^2(\mathbf{x},t)\rangle = \frac{\hbar}{\pi\varepsilon_0 c^3}\int_0^\infty\frac{\mathrm{d}\omega}{2\pi}\omega^3 \tag{2.94}$$

because the integral diverges at the upper limit. This correlates with an infinite electromagnetic energy density in vacuum (multiply with $\varepsilon_0/2$ and add the magnetic component, which doubles the result). This "infinite vacuum energy" is one of the unresolved problems in physics. "Intuitive cutoffs" at short wavelengths, for example, at the Planck scale ($10^{-35}\,\mathrm{m}$) give a finite energy density, but with a value that differs by something like 100 orders of magnitude from the energy density associated with cosmological observations (including "dark energy", "cosmological constants" and so on).

Exercise. Find a cutoff such that the vacuum energy density equals the 'critical density of the Universe' (the critical mass density is roughly $10^{-29}\,\mathrm{g/cm}^3$, given the current expansion rate of the Universe.)

A finite value can be found, if we compute an autocorrelation function of the electric field. A similar calculation gives

$$\langle \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t')\rangle = \int_0^\infty\frac{\mathrm{d}\omega}{2\pi}\frac{\hbar\omega_\kappa}{2\varepsilon_0 c^3}\rho(\omega)\,\mathrm{e}^{-\mathrm{i}\omega(t-t')} \tag{2.95}$$

This leads to the integral representation of the Γ-function in the complex plane and finally to

$$\langle \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t')\rangle = \frac{3\hbar}{\pi^2\varepsilon_0 c^3\,\tau^4} \tag{2.96}$$

which is finite for all $\tau = t - t' \neq 0$. For any finite value of τ, the electric vacuum energy density is thus of the order of $\hbar\omega_\tau/\lambda_\tau^3$ with the characteristic frequency $\omega_\tau = 1/\tau$ and wavelength $\lambda_\tau = c\tau$.

2.2.4 Casimir energy

The Casimir force is the attraction between two metallic mirrors placed in vacuum. It is interpreted in terms of the change in the zero-point energy (the famous $\frac{1}{2}\hbar\omega$ of the harmonic oscillator ground state) induced by the presence of the mirrors. We give here an sketch of the calculation done by Casimir around 1948 [Proc. Kon. Ned. Akad. Wet. **51** (1948) 793].

We consider the ground state energy of the multi-mode electromagnetic field

$$E_0 = \sum_{\mathbf{k}\lambda} \frac{\hbar\omega_{\mathbf{k}\lambda}}{2}$$

that is of course infinite and compare the cases of a planar cavity formed by two mirrors (distance L) and empty space (i.e., two mirrors infinitely apart). In the first case, we have standing wave modes between the mirrors with a frequency

$$\omega^{(cav)} = c\sqrt{K^2 + k_n^2}, \qquad k_n = \frac{n\pi}{L}$$

with $K^2 = k_x^2 + k_y^2$ and $n = 1, 2, \ldots$, while in empty space,

$$\omega = c\sqrt{K^2 + k_z^2}$$

with $-\infty < k_z < \infty$. We first compute the difference in the electromagnetic mode density per volume AL where A is the 'quantization area' in the xy-plane. We cheat with the polarizations and multiply by a factor 2:

$$\begin{aligned} \rho_L(\omega) &= \frac{4\pi}{AL}\sum_{\mathbf{K},n}\delta\Big(\omega - c\sqrt{K^2+k_n^2}\Big) \\ &= \frac{2}{L}\int_0^\infty K\mathrm{d}K\sum_n \delta\Big(\omega - c\sqrt{K^2+k_n^2}\Big) \end{aligned} \tag{2.97}$$

The integration over K can be performed with the substitution $K \mapsto c\sqrt{K^2 + k_n^2}$ and gives

$$\rho_L(\omega) = \frac{2\omega}{Lc^2} \sum_n \Theta(\omega - ck_n) \tag{2.98}$$

where Θ is the step function. It arises because for a given n, there are no modes with frequency smaller than ck_n. The same calculation in the infinite volume gives

$$\rho_\infty(\omega) = \frac{2\omega}{c^2} \int_0^\infty \frac{\mathrm{d}k_z}{\pi} \Theta(\omega - ck_z) \tag{2.99}$$

The k_z integral can of course be performed, but we keep it here to illustrate one of the basic features of the Casimir calculation: the result originates from the difference between a 'discrete spectrum' (the sum over the k_n) and a continuum (the integral over k_z).

The Casimir energy is now found as the difference in vacuum energy *per area* in the space of length L between the mirrors:

$$\Delta E = L \int_0^\Omega \frac{\hbar\omega}{2} \left(\rho_L(\omega) - \rho_\infty(\omega)\right) \tag{2.100}$$

We have introduced an upper cutoff frequency Ω because the integrals are likely to diverge in the UV. One of the mathematical difficulties (that we are not going to discuss here) is to what extent the results depend on the cutoff. At a suitable stage of the calculation, we are going to take the limit $\Omega \to \infty$, of course.

The ω-integrals can be performed before the sum over n (the integral over k_z), and one gets

$$\Delta E = \frac{\hbar}{6\pi c^2} \left[\sum_{n=1}^{\lfloor \Omega L/\pi c \rfloor} \left(\Omega^3 - (ck_n)^3\right) - \int_0^{\Omega/c} \frac{\mathrm{d}k_z}{\pi} \left(\Omega^3 - (ck_z)^3\right) \right] \tag{2.101}$$

where $\lfloor x \rfloor$ is the largest integer smaller than x. Introducing the number $N = \Omega L/\pi c$ and the dimensionless integration variable $z = k_z L/\pi$, this can be written in the form

$$\Delta E = \frac{\hbar c \pi^2}{6L^3} \left[\sum_{n=1}^{\lfloor N \rfloor} \left(N^3 - n^3\right) - \int_0^N \mathrm{d}z \left(N^3 - z^3\right) \right] \tag{2.102}$$

The difference in brackets is some magic number and equal to $-1/120$ in the limit $N \to \infty$. (A proof is sketched below.) The Casimir energy of two mirrors is

thus equal to

$$\Delta E = -\frac{\hbar c \pi^2}{720 L^3} \tag{2.103}$$

so that the force per unit area is $F_C/A = -\hbar c \pi^2/240 L^4$: since the energy decreases as $L \to 0$, the two mirrors placed in vacuum attract each other.

Note that this result is independent of the nature of the mirrors, as well as their electric charge. The electromagnetic field only enters inasmuch as its modes give a contribution to the energy of the vacuum state. Field theorists have computed the contribution to the Casimir energy from the Dirac electron field, for example. It is small if the mirror separation is large compared to the Compton wavelength $\hbar/mc \approx 2.5\,\mathrm{pm}$ — which is nearly always the case. The Casimir energy, being attractive, is sometimes thought of a means to 'stabilize' a classical model of the electron (a bag of charge) against the Coulomb repulsion.

Sum minus integral

We use a trick in the complex plane. There is a theorem for functions f and D that are analytical in a domain limited by the integration contour $\mathcal{C}$:

$$\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \mathrm{d}z\, f(z) \frac{\mathrm{d}}{\mathrm{d}z} \log D(z) = \sum_n f(z_n) \tag{2.104}$$

where the z_n are the zeros of D in the interior of the contour. We will use $f(z) = N^3 - z^3$ and choose $D(z)$ such that it is zero for the values $z_n = n$: $D(z) = \sin(\pi z)$. The differentiation under the integral sign gives

$$\sum_{n=0}^{N} (N^3 - n^3) \;=\; \frac{1}{2} \oint_{\mathcal{C}_N} \mathrm{d}z\, (N^3 - z^3) \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} \tag{2.105}$$

We chose an integration contour as shown in fig. 2.1 running from $+N$ above the real axis to 0 and going back to $+N$ below the real axis (the sum over all positive zeros of $\sin \pi z$ thus gives the sum on the left hand side). We are eventually interested in the limit $N \to \infty$. Make the following transformations on the upper and lower part of the contour:

$$\text{upper part:} \quad \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} = -1 + \frac{2\mathrm{e}^{\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

$$\text{lower part:} \quad \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} = 1 + \frac{2\mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

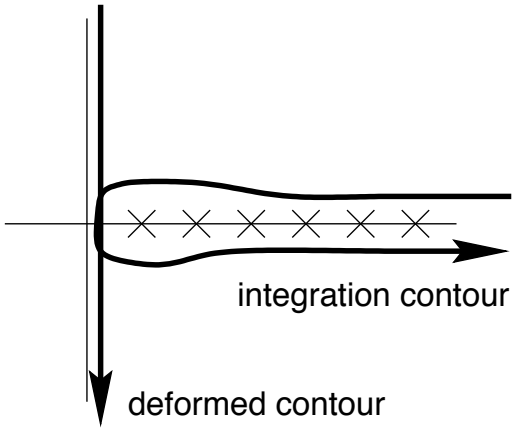

Figure 2.1: Integration contour for (2.105).

The constants ± 1 give for both the upper and lower path an integral over $N^3 - z^3$ that can be combined into

$$\sum_{n=0}^{N}(N^3 - n^3) \;=\; \int_0^N \mathrm{d}z\,(N^3 - z^3) + \oint_{\mathcal{C}} \mathrm{d}z\,(N^3 - z^3)\frac{\mathrm{e}^{\pm \mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

In the second integral, the exponential takes the appropriate sign on the upper and lower parts of the contour. The first integral on the right hand side is exactly the integral that we have to subtract in Eq.(2.102). The upper and lower parts of the contour can now be shifted onto the (positive or negative) imaginary axis because the integrand has no singularities (these are located on the real axis only). The quarter-circle with radius $|z| = N$ contributes only a negligible amount because of the $\mathrm{e}^{\pm \mathrm{i}\pi z}$.

Choosing $z = \pm \mathrm{i}t$ on the imaginary axis, we get

$$\begin{aligned}
&\oint_{\mathcal{C}} \mathrm{d}z\,(N^3 - z^3)\frac{\mathrm{e}^{\pm \mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} \\
&= -\mathrm{i}\int_0^\infty \frac{\mathrm{d}t\,[N^3 - (\mathrm{i}t)^3]\,\mathrm{e}^{-\pi t}}{\mathrm{e}^{-\pi t} - \mathrm{e}^{\pi t}} - \mathrm{i}\int_0^\infty \frac{\mathrm{d}t\,[N^3 - (-\mathrm{i}t)^3]\,\mathrm{e}^{-\pi t}}{\mathrm{e}^{\pi t} - \mathrm{e}^{-\pi t}} \\
&= -2\int_0^\infty \frac{\mathrm{d}t\,t^3}{\mathrm{e}^{2\pi t} - 1}
\end{aligned}$$

Note that the imaginary parts of the two integrals that involve N^3 cancel each other: we have finally eliminated the cutoff.

You have encountered the last integral in the context of blackbody radiation. Changing to the integration variable $t' = 2\pi t$, the integral gives $1/240$, so that

we have in the end

$$\lim_{N\to\infty}\left(\sum_{n=0}^{N}(N^3-n^3)-\int_0^N \mathrm{d}z\,(N^3-z^3)\right)=-\frac{2}{240}=-\frac{1}{120} \tag{2.106}$$

as announced in the text.

2.3 Master equations in quantum optics

2.3.1 Idea

A master equation describes the time evolution of a quantum system beyond the Schrödinger equation. It applies to "open systems" or systems for which the Hamiltonian is not completely known. Master equations are the quantum analogue of kinetic theories that describe the dynamics of a system including different dissipative effects. One can thus follow the approach of the system towards thermal equilibrium.

In quantum optics, the "system" can be an atom, a collection of atoms, or a field mode in a cavity. The system is "open" because it interacts with the "rest of the world", manifest via a continuum of quantized field modes. An atom decays irreversibly by emitting a photon into previously empty vacuum modes. This phenomenon of "spontaneous emission" does not allow for a description in terms of a Hamiltonian. In fact, the state of the atom does not remain pure, and entropy increases because the photon can be emitted into any direction of space. A cavity mode decays because photons escape through the cavity mirrors, effectively becoming excitations of the "modes outside the cavity". Alternatively, the field energy can be absorbed by the material making up the mirrors.

We focus first on the derivation of a master equation for a two-level atom. We shall find from the general theory two results: (1) the rate of spontaneous emission of an electronically excited state and (2) Glauber's formula for the signal of a photodetectors in terms of normally ordered correlation functions of the electric field operator.

Time scales

The derivation builds on the following hierarchy of time scales:

$$\frac{2\pi}{\omega_\mathrm{A}}\leq\tau_\mathrm{c}\ll\frac{2\pi}{\Omega}\sim\frac{1}{\gamma} \tag{2.107}$$

where the shortest time scale is the optical period of the light field that is near-resonant with the atomic transition frequency ω_{A}. This is typically smaller than a few fs (femtoseconds) (in the visible range).

The "correlation time" τ_{c} of the electromagnetic field describes qualitatively the time interval over which the quantum fluctuations of the electromagnetic field (that couple significantly to the atom) are "smooth". Two electric field measurements are not significantly "similar" if they are taken at intervals larger than τ_{c}. We give an estimate below, it turns out to be a few times larger than the optical period.

The third time scale is the Rabi period that scales with the inverse of the Rabi frequency of a laser. This depends of course on the laser power (and the atomic transition dipole), but typical values are in the 1–10 ns (nanoseconds) range. Note that this is much longer than the optical period and the field correlation time.

The last time scale is the lifetime of the excited state of the atom, inversely proportional to the spontaneous decay rate γ. This is typically in the 1–10 ns range as well, depending on the atom.

We thus have a situation similar to "Brownian motion": the atom is "shaken around" by the vacuum field fluctuations that vary much faster than the typical dynamics for the atomic state (Rabi oscillations, spontaneous decay). The master equation takes advantage of this separation of time scales to find an equation of motion that can be applied on "slow time scales". As a result, one gets a single equation that describes both, the "Hamiltonian" effects of a laser (Rabi oscillations) and the "dissipative / friction" effects due to vacuum fluctuations.

Hamiltonian

Two-level atom, field, and electric dipole interaction:

$$H = \frac{\hbar\omega_A}{2}\sigma_3 + \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_k \hbar\left(g_k^* a_k^\dagger \sigma + g_k \sigma^\dagger a_k\right) \tag{2.108}$$

neglect the zero-point energy of the field. Coupling constant $\hbar g_k = -\mathbf{d}_{\mathrm{eg}} \cdot \mathbf{f}_k(\mathrm{x}_A)(\hbar\omega_k/2\varepsilon_0)^{1/2}$ with the mode function $\mathbf{f}_k(\mathrm{x})$ evaluated at the position of the atom. Sometimes called 'vacuum Rabi frequency'.

In the following, we also use the (re-scaled) electric field operator

$$E(t) = \sum_k g_k a_k(t) + \mathrm{h.c.} \tag{2.109}$$

Field correlation time

To get an estimate of the field correlation time, we compute the two-time correlation (or coherence) function

$$C(\tau) = \langle E(t+\tau)E(t)\rangle \tag{2.110}$$

If we use the coupling constants g_k to write the mode expansion of the quantized field (this is equivalent to re-scaling the electric field so that it has units of frequency), we get

$$\begin{aligned}\langle E(t+\tau)E(t)\rangle &= \sum_k |g_k|^2 \langle a_k(t+\tau)a_k^\dagger(t)\rangle \\ &= \sum_k |g_k|^2 \, \mathrm{e}^{-\mathrm{i}\omega_k\tau}\end{aligned} \tag{2.111}$$

An explicit calculation using the mode amplitudes in free space leads to an integral of the form

$$C(\tau) \sim \int_0^\infty \mathrm{d}\omega \, \omega^3 \, \mathrm{e}^{-\mathrm{i}\omega_k\tau} \tag{2.112}$$

which looks like a third derivative of a δ-function in τ. Well, not exactly since the integrations starts at $\omega = 0$. Generalizing to finite temperature, the Fourier transform of the correlation function (2.112) becomes

$$C(\omega) \sim \omega^3 \bar{n}(\omega) = \frac{\omega^3}{\exp(\hbar\omega/k_\mathrm{B}T) - 1} \tag{2.113}$$

The sign convention of the Fourier transform is chosen here such that $\omega > 0$ corresponds to photon numbers $a^\dagger a$ ('normal order'), while at $\omega < 0$, the anti-normal order $aa^\dagger$ is picked. See Fig.2.2: the spectrum is proportional to $-\omega^3$ for negative frequencies, $-\omega \gg k_\mathrm{B}T/\hbar$. Near zero frequency, it has a quadratic behaviour, at positive frequencies, a maximum near $\omega = 2k_\mathrm{B}T/\hbar$ (Wien displacement law) and an exponential decay for $\omega \gg k_\mathrm{B}T/\hbar$.

Actually, we forgot one thing in this estimate: the Hamiltonian we start with is only valid for field modes k near-resonant with the atomic transition, $\omega_k \sim \omega_\mathrm{A}$. It thus seems reasonable to restrict the frequency integral in Eq.(2.112) to an interval around ω_A, with a width $\Delta\omega$ that is typically smaller than ω_A.

Now, from the properties of the Fourier transformation, we know that the correlation function $C(\tau)$ given by the integral (2.112) has a "width in time"

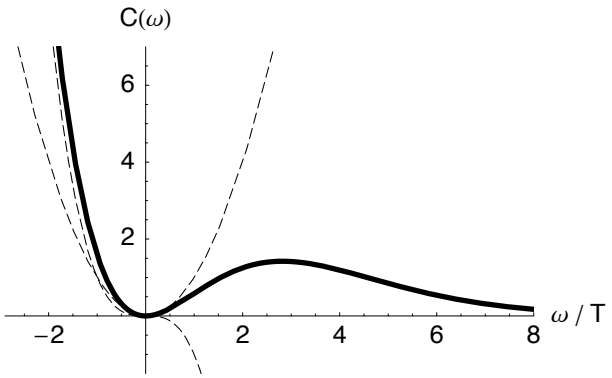

Figure 2.2: Spectrum $C(\omega)$, Eq.(2.113), of the radiation field (normal order) at finite temperature. The frequency is plotted in units of $k_{\rm B}T/\hbar$. The dashed lines are ω^2 and $-\omega^3$.

given by $\Delta\tau\,\Delta\omega \sim 1$. Since this width is precisely the definition of the correlation time, we have

$$\frac{1}{\omega_{\rm A}} \leq \frac{1}{\Delta\omega} \sim \tau_{\rm c}. \tag{2.114}$$

Typical value: a few 10 fs.

2.3.2 Evolution of the atomic operators

We shall work in the Heisenberg picture. Note that operators describing different degrees of freedom (field, atom) commute at equal times.

Atomic dipole

"eliminate" the field operators by solving their equation of motion:

$$\frac{{\rm d}a_k}{{\rm d}t} = \frac{\rm i}{\hbar}\,[H,\,a_k] = -{\rm i}\omega_k a_k - {\rm i}g_k^*\sigma \tag{2.115}$$

$$a_k(t) = a_k(0)\,{\rm e}^{-{\rm i}\omega_k t} - {\rm i}g_k^* \int_0^t {\rm d}t'\,\sigma(t')\,{\rm e}^{-{\rm i}\omega_k(t-t')} \tag{2.116}$$

the "particular solution" contains the "past" of the atomic dipole operator $\sigma(t')$. Insert this into the equation for σ. Let's look at this equation first:

$$\frac{{\rm d}\sigma}{{\rm d}t} = \frac{\rm i}{\hbar}\,[H,\,\sigma] = -{\rm i}\omega_{\rm A}\sigma + {\rm i}\sum_k g_k \left[\sigma^\dagger,\,\sigma\right] a_k$$

$$= \quad -\mathrm{i}\omega_{\mathrm{A}}\sigma + \mathrm{i}\sum_k g_k \sigma_3 a_k \tag{2.117}$$

We see here that the two-level atom leads to nonlinear equations of motion: the operator product $\sigma_3 a_k$ appears. For the moment, these operators commute (at equal times). But we now want to insert the solution (2.116) for $a_k(t)$, and the two terms that appear here do not commute separately with σ_3. For this reason, we take now a specific operator order (so-called "normal order") where the annihilation operators (a_k or σ) act first. This is the order already used in Eq.(2.117), and we thus get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_{\mathrm{A}}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) + \int_0^t \mathrm{d}t' \sum_k |g_k|^2 \mathrm{e}^{-\mathrm{i}\omega_k(t-t')}\sigma_3(t)\sigma(t') \tag{2.118}$$

where we used the abbreviation

$$\mathcal{E}_0(t) = \sum_k g_k a_k(0)\,\mathrm{e}^{-\mathrm{i}\omega_k t} \tag{2.119}$$

for the freely evolving electric field operator (positive frequency component only, and re-scaled by the atomic transition dipole). In the integral over t', we see that the correlation function $C(t-t')$ appears, and using $\tau = t - t'$ as integration variable, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_{\mathrm{A}}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) + \int_0^t \mathrm{d}\tau\, C(\tau)\sigma_3(t)\sigma(t-\tau) \tag{2.120}$$

Now comes the *key observation*: under the time integral occur two very different functions. The correlation function $C(\tau)$ is very narrow in τ. The atomic operator $\sigma(t-\tau)$ contains a "fast free evolution" (generated by the first term in its equation of motion), but once this is factored out, we anticipate that its evolution is "slow":

$$\sigma(t-\tau) = \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{A}}(t-\tau)}\tilde{\sigma}(t-\tau) \approx \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{A}}(t-\tau)}\tilde{\sigma}(t) = \mathrm{e}^{+\mathrm{i}\omega_{\mathrm{A}}\tau}\sigma(t) \tag{2.121}$$

Here, we introduced temporarily the dipole operator $\tilde{\sigma}(t')$ (in an interaction picture).

The main benefit of this approximation is that only atomic operators at time t appear in the equation of motion. The two-level commutation rules[1] give

[1] A pedantic remark: operator products evolve as products under the Heisenberg equations of motion. This is because taking the commutator with a product is compatible with the product rule of (time) derivatives.

$\sigma_3(t)\sigma(t) = -\sigma(t)$, and we end up with

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{A}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) - \sigma(t)\int_0^t \mathrm{d}\tau\, C(\tau)\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{A}\tau} \tag{2.122}$$

The upper integration limit t is actually the difference between our initial time and the actual time where the equation of motion is computed. We now let this time difference be much larger than the correlation time τ_c. This is consistent with the assumption that the atomic dynamics is slow on the scale of the field's correlation time. Then the integrand is effectively zero at the upper limit, and we get a constant complex number

$$\gamma + \mathrm{i}\delta\omega_\mathrm{A} = \int_0^{t\gg\tau_\mathrm{c}} \mathrm{d}\tau\, C(\tau)\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{A}\tau} = \frac{S(\omega_\mathrm{A})}{2} + \mathrm{i}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{S(\omega)}{\omega-\omega_\mathrm{A}} \tag{2.123}$$

where $S(\omega)$ is the Fourier transform of the correlation function $C(\tau)$ and $\mathcal{P}$ means the principal part of the integral.

Spontaneous decay rate and Lamb shift

Explicit calculation: spectrum of vacuum fluctuations

$$S(\omega) = 2\pi\sum_k |g_k|^2\delta(\omega_k-\omega) \tag{2.124}$$

Take a plane wave expansion and sum over the polarization vectors $\mathbf{u}_{\mathbf{k}\lambda}$ in the coupling constants g_k

$$\sum_\lambda |\mathbf{d}_\mathrm{eg}\cdot\mathbf{u}_{\mathbf{k}\lambda}|^2 = |\mathbf{d}_\mathrm{eg}|^2 - |\mathbf{d}_\mathrm{eg}\cdot\hat{\mathbf{k}}|^2 \tag{2.125}$$

where $\hat{\mathbf{k}}$ is the unit vector along $\mathbf{k}$. This formula arises because the $\mathbf{u}_{\mathbf{k}\lambda}$ are perpendicular to $\mathbf{k}$. Integration over the angles of $\mathbf{k}$ gives

$$\int \mathrm{d}\Omega(\hat{\mathbf{k}})\left(|\mathbf{d}_\mathrm{eg}|^2 - |\mathbf{d}_\mathrm{eg}\cdot\hat{\mathbf{k}}|^2\right) = 4\pi|\mathbf{d}_\mathrm{eg}|^2 - \frac{4\pi}{3}|\mathbf{d}_\mathrm{eg}|^2 = \frac{8\pi}{3}|\mathbf{d}_\mathrm{eg}|^2 \tag{2.126}$$

The integral over the length of $\mathbf{k}$ is trivial because of the δ-function in Eq.(2.124), its length is fixed to $|\mathbf{k}| = \omega/c$. Putting everything together, we get

$$S(\omega) = \frac{2\pi}{3\hbar^2}|\mathbf{d}_\mathrm{ge}|^2\frac{\hbar\omega}{2\varepsilon_0}\frac{8\pi\omega^2}{(2\pi c)^3} \tag{2.127}$$

where the last factor is the density of field modes per $\mathrm{d}\omega$ and volume. We can thus say that the spontaneously decaying atom is a "detector" for vacuum field fluctuations. The decay rate γ

$$\gamma = \frac{1}{2} S(\omega_{\mathrm{A}}) = \frac{|\mathbf{d}_{\mathrm{ge}}|^2 (\omega_{\mathrm{A}}/c)^3}{6\pi\hbar\varepsilon_0} \tag{2.128}$$

is also called the "natural linewidth" of the atomic transition $|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ because it gives the width in frequency of the spontaneous emission spectrum. In order of magnitude, with $|\mathbf{d}_{\mathrm{ge}}| \sim e a_0$ (electron charge $\times$ atom size)

$$\frac{\gamma}{\omega_{\mathrm{A}}} \sim \alpha_{\mathrm{fs}} (a_0/\lambda_{\mathrm{A}})^2 \sim \alpha_{\mathrm{fs}}^3 \tag{2.129}$$

with fine structure constant $\alpha_{\mathrm{fs}} = e^2/4\pi\varepsilon_0\hbar c \approx 1/137.04$ and wavelength λ_{A} of resonant transition. Hence, indeed decay is "slow" on the scale of the optical period.

Frequency shift $\delta\omega_{\mathrm{A}}$ is related to asymmetry of vacuum spectrum around the transition frequency. Interpretation from second order perturbation theory: modes slightly below the atomic resonance, $\omega_k \leq \omega_{\mathrm{A}}$ tend to push the level $|\mathrm{e}\rangle$ upwards. Modes above resonance: push downwards. Null effect if spectrum is flat.

Actual calculation requires UV cutoff ω_{uv} and interaction Hamiltonian beyond the resonance (rotating wave) approximation (Hans Bethe $\sim$ 1948, calculation of the Lamb shift). Order of magnitude:

$$\delta\omega_{\mathrm{A}} \approx \gamma \log(\omega_{\mathrm{uv}}/\omega_{\mathrm{A}}) \tag{2.130}$$

and a consistent cutoff is the electron rest mass, $\omega_{\mathrm{uv}} \sim m_{\mathrm{e}} c^2/\hbar$. Hence comparable to the natural linewidth. Note: relativistic theory required, where wavelengths up to electron Compton wavelength $\hbar/m_{\mathrm{e}}c$ are involved. This goes beyond the "long wavelength approximation" because $\hbar/m_{\mathrm{e}}c \sim \alpha_{\mathrm{fs}} a_0 \ll a_0$.

Atomic populations

Actually, we did not yet show that γ is the rate of decay for the excited state. For this, we need the equation of motion for the occupations of the two energy levels. This is described by the atomic operator σ_3, also called the "inversion" because $\langle\sigma_3\rangle > 0$ when the excited state is more occupied than the ground state.

Heisenberg equation of motion

$$\begin{aligned}\frac{\mathrm{d}\sigma_3}{\mathrm{d}t} &= +2\mathrm{i}\sum_k \left[g_k^* a_k^\dagger \sigma - g_k \sigma^\dagger a_k\right] \\ &= 2\mathrm{i}\left[\mathcal{E}_0^\dagger(t)\sigma - \sigma^\dagger \mathcal{E}_0(t)\right] \\ &\quad - 2\int_0^t \mathrm{d}t' \left[C^*(t-t')\sigma^\dagger(t')\sigma(t) + C(t-t')\sigma^\dagger(t')\sigma(t)\right] \quad (2.131)\end{aligned}$$

where we inserted the formal solution for $a_k(t)$ and brought the operator products in normal order. We apply to the t'-integral the same prescription as before and get

$$\begin{aligned}&\int_0^t \mathrm{d}t' \left[C^*(t-t')\sigma^\dagger(t')\sigma(t) + C(t-t')\sigma^\dagger(t')\sigma(t)\right] \\ &\approx (\gamma - \mathrm{i}\delta\omega_\mathrm{A})\sigma^\dagger(t)\sigma(t) + (\gamma + \mathrm{i}\delta\omega_\mathrm{A})\sigma^\dagger(t)\sigma(t) = \gamma(\sigma_3(t) + \mathbb{1}) \quad (2.132)\end{aligned}$$

where the frequency shift drops out. By construction, the operator $\sigma_3(t)+\mathbb{1}$ gives the occupation of the excited state. From Eqs.(2.131, 2.132), we can thus read off the decay rate 2γ for the excited state population, while the ground state remains stable.

Finally, the equation for the inversion operator becomes

$$\frac{\mathrm{d}\sigma_3}{\mathrm{d}t} = -2\gamma(\sigma_3(t) + \mathbb{1}) + 2\mathrm{i}\left[\mathcal{E}_0^\dagger(t)\sigma - \sigma^\dagger \mathcal{E}_0(t)\right] \quad (2.133)$$

Combined with the equation of motion for the atomic dipole operator,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -(\gamma + \mathrm{i}\omega_\mathrm{A})\sigma + \mathrm{i}\sigma_3 \mathcal{E}_0(t) \quad (2.134)$$

we have thus found the "optical Bloch equations".

In Eq.(2.134), we have made the replacement $\omega_\mathrm{A} + \delta\omega_\mathrm{A} \mapsto \omega_\mathrm{A}$ for the atomic frequency. This is called "renormalization": we combine the shift induced by the coupling to the vacuum field with the "naked" transition frequency into the frequency that can be physically observed. Recall that in reality, we can never "switch off" the coupling to the vacuum. Hence the "naked" two-level atom that we started with is actually a theoretical artefact.

2.3.3 Application 1: Bloch equations

Note that we did not specify yet the state of the light field: it is in fact encoded in the operator $\mathcal{E}_0(t)$ that depends on the initial field operators $a_k(0)$. Two examples will be studied now, the first one being an atom driven by a laser field. We have argued that to a good approximation, we can assume that the light field in a coherent state $|\alpha_{\rm L}\rangle$. We assume that at $t = 0$, the total system is in the product state $|\psi(0), \alpha_{\rm L}\rangle$ and take the expectation value of the Bloch equation. This gives the dynamics of the Bloch vector as follows (optical Bloch equations in the proper sense)

$$\frac{{\rm d}s}{{\rm d}t} = -(\gamma + {\rm i}\omega_{\rm A})s + ({\rm i}/2)s_3\Omega\,{\rm e}^{-{\rm i}\omega_{\rm L}t} \tag{2.135}$$

$$\frac{{\rm d}s_3}{{\rm d}t} = -2\gamma(s_3(t)+1) + {\rm i}\left[\Omega^*(t){\rm e}^{{\rm i}\omega_{\rm L}t}s - s^*\Omega{\rm e}^{-{\rm i}\omega_{\rm L}t}\right] \tag{2.136}$$

where $\Omega/2 = \langle\alpha_{\rm L}|\mathcal{E}_0(0)|\alpha_{\rm L}\rangle$ is the (complex) Rabi frequency and $\omega_{\rm L}$ the frequency of the laser mode.

These equations have time-dependent coefficients, but this can be removed by making a transformation into a "rotating frame". We make the replacement

$$s(t) \mapsto s(t){\rm e}^{-{\rm i}\omega_{\rm L}t} \tag{2.137}$$

where the "new" $s(t)$ satisfies Bloch equations with time-independent coefficients

$$\boxed{\begin{aligned}\frac{{\rm d}s}{{\rm d}t} &= -(\gamma - {\rm i}\delta)s + ({\rm i}/2)s_3\Omega \\ \frac{{\rm d}s_3}{{\rm d}t} &= -2\gamma(s_3(t)+1) + {\rm i}\left[\Omega^*s - s^*\Omega\right]\end{aligned}} \tag{2.138}$$

where $\delta = \omega_{\rm L} - \omega_{\rm A}$ is the "laser detuning". Note that the symbols are not the same throughout the books: the Rabi frequency Ω or the decay rate γ can differ by a factor of 2 (or -2), the detuning can have the opposite sign.

The Bloch equations are a "workhorse" of atomic physics and quantum optics. They are used to compute light absorption, excitation spectra, population transfer, radiation forces on atoms etc. In the exercises, you compute the stationary state of the Bloch equations (attention with the different sign for δ: this one should be correct)

$$s_{\rm ss} = \frac{-{\rm i}(\Omega/2)(\gamma + {\rm i}\delta)}{\gamma^2 + \delta^2 + \Omega^2/2} \tag{2.139}$$

$$s_{3,{\rm ss}} = \frac{-(\gamma^2 + \delta^2)}{\gamma^2 + \delta^2 + \Omega^2/2} \tag{2.140}$$

Discussion: average atomic dipole operator (induced by laser field), average inversion. Line broadening.

Exercise: total excitation N, does not commute when laser included. Expectation value of $\dot{N}$ in stationary state, interpretation as total scattered intensity.

Exercise: spectrum of spontaneous emission, from formal solution. Need formal solution of atomic dipole operator, Eq.(2.145).

2.3.4 Application 2: the Glauber photodetector

Roy Glauber (Nobel prize 2005) developped in the 1960s the theory of photodetection. His main result is that the signal of a photodetector is proportional to

$$I(t) \propto \langle E^{(-)}(t) E^{(+)}(t) \rangle \tag{2.141}$$

where $E^{(+)}(t)$ is the positive frequency part of the electric field operator at the detector position. This signal is constructed in such a way that if the field is in the vacuum state, a detector gives no signal: perfectly reasonable. But due to the presence of vacuum fluctuations (also nonzero expectation values of a product of field operators!), not easy to implement in the theory.

We can recover the relevant features of Glauber's theory with our two-level atom. We shall actually show that under suitable approximations,

$$I(t) \propto \langle \mathcal{E}_0^\dagger(t) \mathcal{E}(t) \rangle \tag{2.142}$$

where $\mathcal{E}(t)$ is the re-scaled electric field operator we introduced in Eq.(2.119). Note that it contains positive frequency components only, and also only those modes that are near-resonant with the atomic transition frequency $\omega_{\rm A}$. Indeed, Glauber's model for a photodetector is a two-state system that is prepared in the ground state. Incident light can be absorbed, leading to some population in the excited state. This population is then "rapidly removed" from the system. A physical example: the excited atom is ionized and the free electron moves away (it cannot come back to recombine into the ground state). This is actually the process that happens in a photomultiplier ("avalanche photodiode").

So let us see what our Heisenberg equations (2.133, 2.134) give when the atom in the ground state and the field is in an arbitrary state. We are interested in the rate of change of the excited state population:

$$I := \frac{\mathrm{d}p_e}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \frac{\sigma_3 + \mathbb{1}}{2} \rangle = \frac{1}{2} \frac{\mathrm{d}\langle \sigma_3 \rangle}{\mathrm{d}t} \tag{2.143}$$

Since the atom is in the ground state, the operator $\sigma_3 + \mathbb{1}$ that appears in Eq.(2.133) averages to zero. We are left with

$$I = \mathrm{i}\langle \mathcal{E}_0^\dagger(t)\sigma(t) - \sigma^\dagger(t)\mathcal{E}_0(t)\rangle \tag{2.144}$$

and insert the formal solution for the atomic dipole operator [similar to Eq.(2.116)]:

$$\sigma(t) = \sigma(0)\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})t} + \mathrm{i}\int_0^t \mathrm{d}t'\,\sigma_3(t')\mathcal{E}_0(t')\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})(t-t')} \tag{2.145}$$

This gives

$$\begin{aligned} I &= -\int_0^t \mathrm{d}t' \left[\langle \mathcal{E}_0^\dagger(t)\sigma_3(t')\mathcal{E}_0(t')\rangle\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})(t-t')} + \mathrm{h.c.}\right] \\ &= \int_0^t \mathrm{d}\tau \left[\langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t-\tau)\rangle\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})\tau} + \mathrm{h.c.}\right] \end{aligned} \tag{2.146}$$

where in the second line, we used the approximation that the inversion $\sigma_3(t')$ is evolving slowly and took its expectation value in the initial state (atom in the ground state, not correlated with the field). If we also assume that the expectation value evolves slowly with time t, $\langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t-\tau)\rangle \approx \langle \mathcal{E}_0^\dagger(t+\tau)\mathcal{E}_0(t)\rangle$ [this is strictly true for a stationary field, but stationary fields do not give not very interesting signals on a photodetector], we can combine the '$+\mathrm{h.c.}$' term into the single integral

$$I = \int_{-t}^t \mathrm{d}t' \langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t+\tau)\rangle\,\mathrm{e}^{-\gamma|\tau|+\mathrm{i}\omega_\mathrm{A})\tau} \tag{2.147}$$

We observe that the photodetector signal is similar to the Fourier transform of the two-time field correlation function. We can already confirm that the signal is given by a normally ordered expectation value of field operator. Let us consider two limits.

Narrow-band detector

If the decay time $1/\gamma$ is "very long" (compared to the correlation time of the field), the photodetector signal is essentially given by the field spectrum, taken at the transition frequency ω_A. Indeed, the τ-integral in Eq.(2.147) picks from the positive frequency operator $\mathcal{E}_0(t+\tau)$ those components that evolve like $\mathrm{e}^{-\mathrm{i}\omega_\mathrm{A})\tau}$.

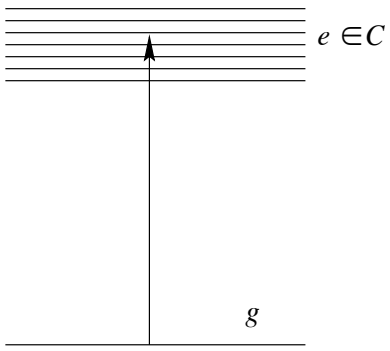

Figure 2.3: Two-level model for a fast photodetector. The signal corresponds to a sum of transition rates into all components of the upper state (quasi)continuum.

A narrow band photodetector is thus simply a 'spectrometer', and the quantity it measures is the Fourier transform of the temporal correlation (or coherence) function. Experimentally, this situation can be achieved by placing a narrow-band frequency filter in front of a photodetector.

Fast detector

In some photodetectors, the excited state has a very short lifetime $1/\gamma$. This happens for example when the excited state actually ionizes and gives away its electron. The excitation rate I is then actually an integral over all components of the excited state 'continuum', as sketched in Fig.2.3. Under the integral, the exponential $\mathrm{e}^{-\gamma|\tau|}$ becomes very short-ranged in τ, and we can make the replacement

$$\mathrm{e}^{-\gamma|\tau|} \mapsto \frac{2}{\gamma}\delta(\tau) \tag{2.148}$$

This leads to Glauber's formula

$$I(t) = \eta\langle\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t)\rangle \tag{2.149}$$

where the expectation value of the instantaneous intensity operator $\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t)$ appears, in normal order, of course. The prefactor η actually is a number characteristic for the detector and is called "quantum efficiency". It must be determined experimentally. If the intensity is scaled to "photons per second", then η gives the detection probability per photon, and I the "rate of detected photons per second".

2.3.5 Application 3: intensity correlations

Problem: "resonance fluorescence" = two-level atom driven by a laser field. Measurement: emitted light intensity $\langle I(t)\rangle$ and its correlations $\langle I(t+\tau)I(t)\rangle$.

Stationary regime, $I_{\rm ss}$, normalized correlation function

$$g^{(2)}(\tau) = \frac{\langle I(t+\tau)I(t)\rangle}{(I_{\rm ss})^2} \tag{2.150}$$

Properties $g^{(2)}(\tau) \geq 0$ because intensity is positive. In the classical theory, $g^{(2)}(0) \geq 1$ because average of square, $\langle I^2\rangle \geq \langle I\rangle^2$. If decorrelation at large times, $g^{(2)}(\infty) = 1$.

Cauchy-Schwarz: $0 \leq g^{(2)}(\tau) \leq g^{(2)}(0)$, will be violated in the quantum theory: "non-classical light".

Classical, chaotic light

Classical treatment, chaotic light (n randomly phased emitters, Poisson statistics with mean value $\bar{n}$)

$$\mathcal{E} = \mathcal{E}_0 \sum_{\alpha=1}^{n} \mathrm{e}^{\mathrm{i}\varphi_\alpha} \tag{2.151}$$

average intensity

$$\langle|\mathcal{E}|^2\rangle = |\mathcal{E}_0|^2 \sum_{\alpha,\beta} \langle \mathrm{e}^{\mathrm{i}\varphi_\alpha - \mathrm{i}\varphi_\beta}\rangle = |\mathcal{E}_0|^2 \sum_{\alpha} = n|\mathcal{E}_0|^2 \tag{2.152}$$

only terms with $\alpha = \beta$ contribute to the double sum, the others average to zero because the relative phases are completely random.

Intensity variance, pick pairs of indices with equal phases

$$\begin{aligned} \langle|\mathcal{E}|^4\rangle &= |\mathcal{E}_0|^4 \sum_{\alpha,\beta,\gamma,\delta} \langle \mathrm{e}^{\mathrm{i}\varphi_\alpha - \mathrm{i}\varphi_\beta + \mathrm{i}\varphi_\gamma - \mathrm{i}\varphi_\delta}\rangle \\ &= |\mathcal{E}_0|^4 \Big(\sum_{\alpha} + \sum_{\substack{\alpha=\beta \\ \neq\gamma=\delta}} + \sum_{\substack{\alpha=\delta \\ \neq\beta=\gamma}}\Big) \\ &= |\mathcal{E}_0|^4 (n + 2n^2) \end{aligned} \tag{2.153}$$

Hence for the normalized correlation function

$$g^{(2)}(0) = \frac{2\overline{n^2} - \overline{n}}{(\overline{n})^2} = 2 + \frac{1}{\bar{n}} \tag{2.154}$$

Average of number of emitters with Poisson statistics assumed in the last step. In the limit $\bar{n} \gg 1$: "bunching" $g^{(2)}(0) = 2$, larger than the asymptotic (decorrelation) value $g^{(2)}(\infty) = 1$. Sketch. This will be violated in the quantum theory of a single emitter: "anti-bunching". The amount of violation is actually a measure of how many quantum emitters contribute to the signal.

Two-level emitter

Quantum treatment. Interpret as two-time joint detection event, define via normal order (absorb one photon at time t, another one at $t + \tau$):

$$g^{(2)}(\tau; t) = \frac{\langle \mathcal{E}^\dagger(t)\mathcal{E}^\dagger(t+\tau)\mathcal{E}(t+\tau)\mathcal{E}(t)\rangle}{\langle \mathcal{E}^\dagger(t)\mathcal{E}(t)\rangle^2} \tag{2.155}$$

Link to atomic dipole operator, assuming that the laser field (the freely evolving field operator) does not contribute to the detected signal:

$$g^{(2)}(\tau; t) = \frac{\sigma^\dagger(t)\sigma^\dagger(t+\tau)\sigma(t+\tau)\sigma(t)\rangle}{\langle \sigma^\dagger(t)\sigma(t)\rangle^2} \tag{2.156}$$

actually, the time argument is shifted by retarded light path r/c between atom and detector. This is a common shift for all time arguments, however, and does not appear in the final result that only depends on time differences (in the steady state).

Direct calculation using the two-level annihilation operators: $g^{(2)}(0) = 0$ because $\sigma(t)\sigma(t) = 0$ ('nilpotent' ladder operator if only two states on the ladder). Hence "anti-bunching" $g^{(2)}(0) < g^{(2)}(\infty)$. Interpretation with photon emission and excited state: after the emission of the first photon, the atom is in the ground state and needs some time to get excited for the next emission. (Note: one also sees why two or more atoms in the detection volume can "fill the antibunching hole".)

Result for the τ-dependence in the stationary regime:

$$g^{(2)}(\tau) = 1 - \mathrm{e}^{-3\gamma\tau/2}\left(\cos\Omega'\tau + \frac{\gamma}{\Omega'}\sin\Omega'\tau\right) \tag{2.157}$$

with $\Omega'^2 + \gamma^2/4 = \Omega^2$ with Ω the laser Rabi frequency. See the Figure for weak and strong drive. **Note:** this signal is fairly easy to measure: one just has to compute the correlation function of the detected intensity with a good time resolution. The τ-dependence (in particular for a weak drive, $\Omega \ll \gamma$) provides a way to measure the excited state lifetime $1/\gamma$.

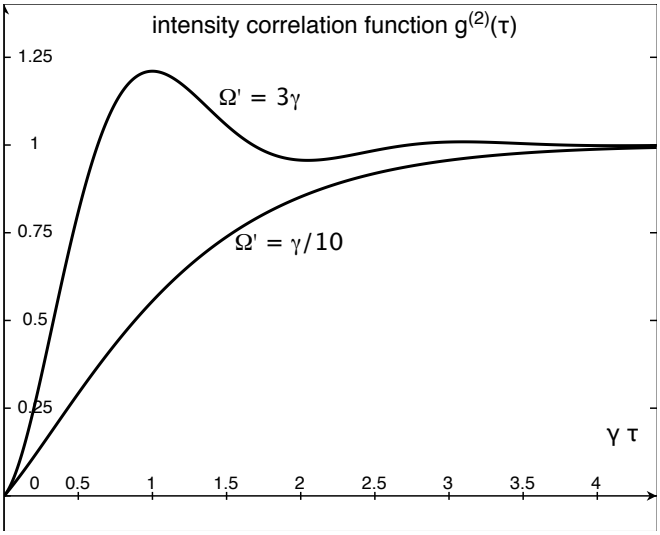

Figure 2.4: Normalized intensity correlation function for the fluorescence of a single two-level atom. The curves for $\tau < 0$ are unphysical.

Sketch origin of this formula: another quantum regression formula. Work out coupled equations for F, G, H with

$$\begin{aligned} g^{(2)}(\tau;t) &\propto \langle \sigma^\dagger(t)\sigma^\dagger(t+\tau)\sigma(t+\tau)\sigma(t)\rangle && (2.158)\\ &= \frac{1}{2}\left(p_{\rm e}(t) + G(\tau;t)\right) && (2.159)\\ F(\tau;t) &= \langle \sigma^\dagger(t)\sigma(t+\tau)\sigma(t)\rangle && (2.160)\\ G(\tau;t) &= \langle \sigma^\dagger(t)\sigma_3(t+\tau)\sigma(t)\rangle && (2.161)\\ H(\tau;t) &= \langle \sigma^\dagger(t)\sigma^\dagger(t+\tau)\sigma(t)\rangle && (2.162) \end{aligned}$$

differential equation for the τ-derivative: take the operator-valued Bloch equations (2.133, 2.134), multiply with the operators at time t and take the average. Key assumption (approximation): for these multi-time correlation functions, the master equation also holds. Initial values from operator products, $G(0;t) = -p_{\rm e}(t)$, $F(0) = 0 = H(0)$.

Interpretation of intensity correlation function, identifying at time t Schrödinger and Heisenberg picture and assuming the atom to be in the steady state $\rho(t) = \rho_{\rm ss}$ [$\pi_{\rm e} = \sigma^\dagger\sigma$ is the projector onto the excited state]

$$\begin{aligned} g^{(2)}(\tau) &\propto {\rm tr}\left[\sigma^\dagger(t)\pi_{\rm e}(t+\tau)\sigma(t)\rho_{\rm ss}\right]\\ &= {\rm tr}\left[\pi_{\rm e}(t+\tau)|{\rm g}\rangle\langle{\rm e}|\rho_{\rm ss}|{\rm e}\rangle\langle{\rm g}|\right] \end{aligned}$$

$$= \quad \mathrm{tr}\,(\pi_{\mathrm{e}}(t+\tau)|\mathrm{g}\rangle\langle\mathrm{g}|)\,p_{\mathrm{e}}(\mathrm{ss}) \tag{2.163}$$

In words: the probability to find the atom in the excited state (again) at time $t+\tau$ after starting at t in the ground state. Note: this interpretation is particularly easy for intensity correlations. For the dipole correlation function,

$$g^{(1)}(\tau) = \langle\sigma^\dagger(t+\tau)\sigma(t)\rangle \tag{2.164}$$

this does not work because it involves the "skew" operator $|\mathrm{g}\rangle\langle\mathrm{e}|$ at the initial time t. This is, of course, not a proper density operator.

We have seen here an example of a quantum regression formula[2]: two-time averages depend on the time difference τ in the same way as one-time averages depend on time t. This actually works only in the stationary regime and in the memoryless case. It is a topic of current research whether a quantum regression formula holds for "master equations with memory" (de Vega & Alonso, 2006; Budini, 2008).

2.4 Cavity QED

So far, we have worked out the dynamics of an atom coupled to a continuum of quantized field modes. We now consider the opposite case where a single field mode dominates the atom-field interaction. This is part of the domain of 'cavity QED'. The name is chosen because a single field mode can be isolated experimentally using cavities with highly reflecting mirrors. In practice, even a cavity sustains many modes (with discrete frequencies, however). We again have to invoke a resonance approximation to single out one mode (that is closest in frequency to an atomic resonance).

2.4.1 Jaynes–Cummings–Paul model

The Hamiltonian for a two-level atom coupled to a single mode has the following simple form

$$H = \frac{\hbar\omega_A}{2}\sigma_3 + \hbar\omega a^\dagger a + \hbar g\left(a^\dagger\sigma + \sigma^\dagger a\right) \tag{2.165}$$

it is called the 'Jaynes–Cummings–Paul model'[3]. The first term is the energy of the atom with Bohr frequency ω_A, the second term the cavity mode energy with

[2] Another example is the calculation of the emission spectrum of resonance fluoresence.

[3] Harry Paul, for many years leader of the group on non-classical radiation at Humboldt Universität zu Berlin, (East) Germany.

the zero-point energy subtracted, the third term the coupling between the two, characterized by a single coupling constant g. In typical experimental setups, $g = g(t)$ is time-dependent and describes how an atom moves spatially in and out of the cavity mode function.

Time evolution

If the light field is described as a single quantized mode, an additional feature occurs in the Rabi oscillations. The key point is that the coupling Hamiltonian, $g(a^\dagger\sigma + \sigma^\dagger a)$, now couples the states $|\mathrm{g}, n\rangle$ and $|\mathrm{e}, n-1\rangle$ where n is the photon number. These states are split (on resonance) in energy by the 'Rabi splitting' $g\sqrt{n}$. Recall that this splitting was $|\Omega|$ for a classical laser field, proportional to the field amplitude. This is mimicked by the scaling with $\sqrt{n}$ since the photon number n is proportional to the field intensity. A special role plays the state $|\mathrm{g}, 0\rangle$: it has no partner with the atom excited and gives the ground state of the JCP-model.

The solution to the Schrdinger equation can be copied from the results of the first chapter, Eq.(1.30), for an initial condition of the form $|\mathrm{g}, n\rangle$ (with $n \geq 1$):

$$|\psi(t)\rangle = \cos(gt\sqrt{n})|\mathrm{g}, n\rangle - \mathrm{i}\sin(gt\sqrt{n})|\mathrm{e}, n-1\rangle \tag{2.166}$$

This applies for exact resonance $\omega = \omega_A$ between field mode and atom. If the detuning $\Delta = \omega - \omega_A$ is nonzero, one gets a more complicated expression where the 'effective Rabi frequency' $\Omega_n = \sqrt{\Delta^2 + 4(gt)^2 n}$ appears.

Exercise. Calculate the time evolution for an initially excited atom.

2.4.2 Collapse and revival

In each sub-space spanned by $|\mathrm{g}, n\rangle$ and $|\mathrm{e}, n-1\rangle$, the system thus performs Rabi oscillations with a slightly different frequency. If one starts with field state that contains many different photon numbers (for example a coherent state), the Rabi oscillations will still evolve at a mean frequency $\approx g\sqrt{\langle n\rangle}$, but at large times, the oscillations will 'get out of phase' because of their frequency spread. This leads to a 'collapse' of the Rabi oscillation amplitude, as illustrated in Figure 2.5. This collapse happens on the time scale $1/g$ which is, for a coherent state, a factor $\sqrt{\langle n\rangle}$ times longer than the period of the initial Rabi oscillations. At still larger times, of order $\sqrt{\langle n\rangle}/g$, the amplitude of the oscillations 'revives' again. This is due to the fact that the Rabi frequencies form a discrete set.

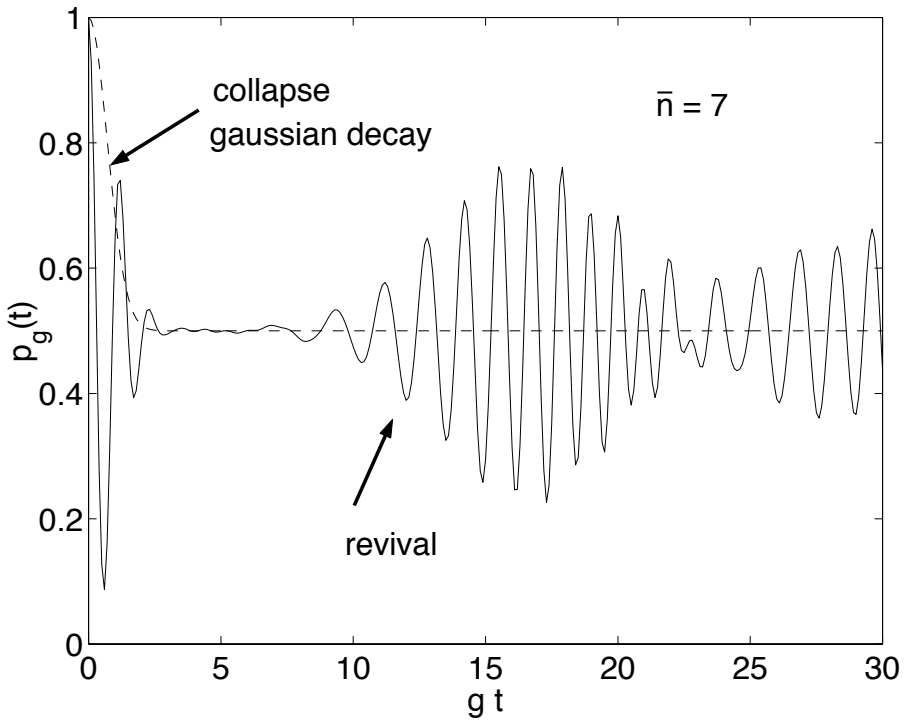

Figure 2.5: Ground state occupation $p_g(t)$ for a two-level atom coupled to a single mode, initially in the coherent state $|\alpha\rangle$ with $|\alpha|^2 = 7$ (= average photon number). Time is in units of the 'single-photon Rabi frequency' g.

Coherent field state

Let us look in more detail on the coherent state to understand this behavior. Coherent states can be introduced already in elementary quantum mechanics for the harmonic oscillator: they are eigenstates of the annihilation operator $a \sim x + \mathrm{i}p$:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{2.167}$$

where $\alpha \in \mathbb{C}$ is a complex eigenvalue. A simple calculation shows the following expansion in the number state basis

$$|\alpha\rangle = \sum_n \mathrm{e}^{-|\alpha^2|/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{2.168}$$

Coherent state are not stationary, but they evolve in time in a simple way: they remain coherent with a different parameter

$$|\alpha\rangle \mapsto |\alpha(t)\rangle = |\alpha\, \mathrm{e}^{-\mathrm{i}\omega t}\rangle \tag{2.169}$$

where ω is the mode frequency. In the complex plane, the motion is thus similar to the classical phase space plane where a harmonic oscillator also rotates. The average photon number remains constant and equal to

$$\langle\alpha(t)|\hat{n}|\alpha(t)\rangle = \langle\alpha(t)|a^\dagger a|\alpha(t)\rangle = |\alpha|^2 \tag{2.170}$$

Coherent states, being eigenstates of a non-hermitean operator, do not form a complete set, but an overcomplete one. (They are not mutually orthogonal.) Note also that eigenstates of the creation operator $a^\dagger$ do not exist.

The '*photon statistics*' of a coherent state is the probability p_n of finding n photons, hence

$$p_n(\alpha) = |\langle n|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha^2|}\frac{|\alpha|^{2n}}{n!} \tag{2.171}$$

This distribution is called the Poisson statistics. Its mean value is $\langle n\rangle_\alpha = |\alpha|^2$, and the variance

$$(\Delta n)^2_\alpha = |\alpha|^2 = \langle n\rangle \tag{2.172}$$

The relative width $\Delta n/\langle n\rangle$ thus scales like $\langle n\rangle^{-1/2}$ and becomes narrow as the average photon number grows.

JCP dynamics

Let us compute as for the classical Rabi oscillations the probability of finding the atom in the ground state. This is given by the sum over all photon numbers

$$p_{\mathrm{g}}(t) = \sum_n |\langle \mathrm{g}, n|\psi(t)\rangle|^2 \tag{2.173}$$

The state vector is given by, using Eqs.(2.166, 2.168),

$$|\psi(t)\rangle = \sum_n \mathrm{e}^{-|\alpha^2|/2}\frac{\alpha^n}{\sqrt{n!}}\Big(\cos(gt\sqrt{n})|\mathrm{g}, n\rangle - \mathrm{i}\sin(gt\sqrt{n})|\mathrm{e}, n-1\rangle\Big) \tag{2.174}$$

We thus find from Eq.(2.173),

$$\begin{aligned} p_g(t) &= \sum_n \mathrm{e}^{-|\alpha^2|}\frac{|\alpha|^{2n}}{n!}\cos^2(gt\sqrt{n}) \\ &= \frac{1}{2} + \frac{1}{2}\sum_n \mathrm{e}^{-|\alpha^2|}\frac{|\alpha|^{2n}}{n!}\cos(2gt\sqrt{n}) \end{aligned} \tag{2.175}$$

where the second term gives the oscillating population we are familiar with from the Rabi flopping. The photon number enters via the scaling of the coupling: the frequency $2g\sqrt{n}$ plays the role of the classical Rabi frequency.

Note that if we start with a one-photon field at resonance, we reach the state $|e;0\rangle$ after some interaction time. We could therefore also take the atom in the excited state and the field in the vacuum state – and find the so-called 'vacuum Rabi oscillations'. This is impossible in the classical theory. One says sometimes that the vacuum fluctuations 'stimulate' the atom to emit a photon. This argument lies at the heart of the interpretation of spontaneous emission in terms of the interaction with the quantized radiation field. The typical exponential decay of the excited state, is however impossible to describe in this simple single-mode model. One needs a mode continuum for that, as we have seen in Section **??**.

The coherent state does not contain a definite number of photons – so what happens to the Rabi oscillations? We calculate now that their amplitude gets damped ('collapse') and may be re-born for some later time (provided no other loss processes occur in the mean time).

Collapse

Eq.(2.175) for the population $p_{\rm g}(t)$ looks like a difficult sum because of the square root in the cosine. Let us therefore assume that our coherent state has a large mean photon number, $\langle n\rangle = |\alpha|^2 \gg 1$. Its relative fluctuations in the photon number are then small, and we may expand the square root around the mean photon number $\bar{n}$:

$$2g\sqrt{n} \approx \bar{\Omega} + \frac{g(n-\bar{n})}{\sqrt{\bar{n}}} + \mathcal{O}((n-\bar{n})^2) \tag{2.176}$$

where the first term is the 'average Rabi frequency' $\bar{\Omega} = 2g|\alpha| = 2g\sqrt{\bar{n}}$. We would like to replace the sum over n by an integral which is easier to solve. This is justified if the interaction time t is sufficiently short so that

$$\frac{gt}{\sqrt{\bar{n}}} \ll 1.$$

In this limit, the argument of the cos changes little from one photon number to the other (on the scale given by the period 2π of the cos), and the sum can be seen as the Riemann sum approximation to an integral.

For the integral, we approximate the photon probability distribution $|c_n|^2$ by a gaussian centred at $\bar{n}$ with variance $\bar{n}$ (since this is the photon number variance for a coherent state) and get

$$p_{\rm g}(t) \approx \frac{1}{2} + \frac{1}{2}\int \frac{{\rm d}n}{\sqrt{2\pi\bar{n}}}\, {\rm e}^{-(n-\bar{n})^2/2\bar{n}} \cos(\bar{\Omega}t + 2g^2 t(n-\bar{n})/\bar{\Omega})$$

$$= \frac{1}{2} + \frac{1}{2}\cos(\bar{\Omega}t)\mathrm{e}^{-(2g^2t/\bar{\Omega})^2\bar{n}/2}$$

We thus see that the Rabi oscillations are damped with a gaussian factor

$$\mathrm{e}^{-(g^2t/g\sqrt{\bar{n}})^2\bar{n}/2} = \mathrm{e}^{-(gt)^2/2} \tag{2.177}$$

on a timescale $t \sim 1/g$ given by the single-photon Rabi frequency. When the coherent state contains a large (average) number of photons, this time scale is much longer than the Rabi period itself.

Revival

At longer times, we cannot replace the sum by an integral because the number states do not give Rabi phases $gt\sqrt{n}$ that are close together: the $\cos$ in Eq.(2.175) varies rapidly from one term in the sum over n to another. To find the first instant of where something different may happen, we consider the time where the coefficient of $n - \bar{n}$ in the expansion (2.176) is such that adjacent photon numbers have Rabi phases that differ by an integer multiple of 2π: the functions $\cos gt\sqrt{n}$ then add up constructively, and we get a 'revival' of the Rabi oscillation amplitude. This happens for the first time at the time given by

$$2\pi = 2gt\sqrt{n+1} - 2gt\sqrt{n} \approx \frac{gt}{\sqrt{\bar{n}}} \quad \Rightarrow t_{\mathrm{rev}} = 2\pi\frac{\sqrt{\bar{n}}}{g},$$

for large $\bar{n}$, this happens way beyond the 'short time regime' we considered before.

To summarize, we have found the following hierarchy of timescales:

$gt \sim 1/\sqrt{\bar{n}}$	$gt \sim 1$	$gt \sim \sqrt{\bar{n}}$
Rabi flopping	collapse	revival

For strong coherent fields like laser beams ($\bar{n} \gg 1$), the revival regime is difficult to observe because the timescale is so long that other effects arise (phase fluctuations of the laser, e.g.). Collapse and revival experiments are therefore most easily done with few photon states (but with somewhat more photons than the few-photon experiments described before). Do not forget that the field starts in a coherent state that is easily prepared.

The exact summation of the ground state probability (2.175) is shown in fig. 2.5 for a mean photon number $\bar{n} = 7$. Although the timescales are not well separated, the gaussian decay gives a good approximation to the collapse. The

revival does not revive the Rabi oscillation with unit amplitude because of the breakdown of the expansion (2.176). Further revivals can be expected at integer multiples of t_{rev}, but higher order terms in the sum are change their shape. (A recent analysis can be found in Karatsuba & Karatsuba, *J Phys A* (2009).)

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