Quanten-Informatik und theoretische Quantenoptik II

Sommersemester 2010

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Problem set 2 Hand out: 30 April 2010 hand in: 17 Mai 2010

Problem 2.1 – Some algebraic properties of Lindblad operators (5 points)

(a) Show that for a two-level system, the Lindblad superoperator

$$\mathcal{L}(\rho) = \gamma L \rho L^\dagger - \frac{\gamma}{2} \left\{ \rho L^\dagger L + L^\dagger L \rho \right\} \tag{2.1}$$

commutes with the unitary operator $U = \exp(\mathrm{i}\theta\sigma_3)$ in the following sense

$$U \mathcal{L}(\rho) U^\dagger = \mathcal{L}(U \rho U^\dagger) \tag{2.2}$$

for all three choices of Lindblad operators $L = \sigma$, $\sigma^\dagger$, σ_3.

(b) You know that the operators $\sigma, \sigma^\dagger, \sigma_3$ form a closed (Lie-)Algebra in the sense that the commutator $[A, B]$ of any two $A, B \in \{\sigma, \sigma^\dagger, \sigma_3\}$ is a linear combination of these operators. Is this still true for the “dissipative algebra” (with $L = \sigma$)

$$(L, L^\dagger, A) \mapsto [L^\dagger,\, A]L + L^\dagger[A,\, L] \tag{2.3}$$

that appears in a positive map? What happens for a harmonic oscillator with $L = a$ if you take as observable algebra the one generated by a, $a^\dagger$? Or by a, $a^\dagger$, $a^\dagger a$?

Answer for the spin: the dissipative algebra is not closed, one has to augment it by one operator. Which one?

Problem 2.2 – Decoherence of two-level systems (5 points)

(a) Compare the ground state of the atom-field Hamiltonian

$$H_{\mathrm{AL}} = \frac{\hbar(\omega_A - \omega_L)}{2}\sigma_3 + \frac{\hbar}{2}(\Omega^* \sigma + \Omega \sigma^\dagger) \tag{2.4}$$

with the stationary solution of the Bloch equations obtained in the last problem set. Why is there a difference?

(b) Consider a two-level system prepared in a superposition state $|\psi(0)\rangle = (|\mathrm{g}\rangle + |\mathrm{e}\rangle)/\sqrt{2}$. Show that $\langle\sigma(0)\rangle := \langle\psi(0)|\sigma|\psi(0)\rangle$ is not zero and argue that this quantity characterizes the “degree of superposition” of the state. Calculate from the Bloch equations (no laser field, for simplicity) the time evolution $\langle\sigma(t)\rangle$ and compare to the spontaneous decay of $p_e(t) = \mathrm{tr}(|\mathrm{e}\rangle\langle\mathrm{e}|\rho(t))$.

Problem 2.3 – Repeated measurements (5 points)

Consider the following positive map for a two-level system

$$T(\rho) = p\rho + q\sigma_3\rho\sigma_3 \tag{2.5}$$

Fix the coefficients p and q by trace conservation. Calculate the n-fold iteration $T^n(\rho)$ and the expectation values $\langle\sigma_3\rangle_n = \mathrm{tr}[\sigma_3 T^n(\rho)]$ and $\langle\sigma\rangle_n$. Interpretation?

Problem 2.4 – Kraus-Stinespring theorem (5 points)

(a) Show that a map in Kraus-Stingspring form,

$$T(\rho) = \sum_k \Omega_k \rho \Omega_k^\dagger, \qquad \sum_k \Omega_k^\dagger \Omega_k = \mathbb{1} \tag{2.6}$$

is completely positive.

(b) Show that the operators Ω_k constructed in the proof of the Kraus-Stinespring theorem satisfy the "completeness relation" (2nd equation in Eq.(2.6)).