Quanten-Informatik und theoretische Quantenoptik II

Sommersemester 2010

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Problem set 6 Hand out: 09 Jul 2010 hand in: 19 Jul 2010

Problem 6.1 – Phase-space distribution for a laser (7 points)

Consider a quantum-optical Fokker-Planck equation for a single-mode laser of the form

$$\partial_t P(r, \theta, t) + \partial_r (G(r^2) - \kappa) r P = \partial_r G(r^2) \partial_r P + \frac{1}{r^2} \partial_\theta^2 P \tag{6.1}$$

where $\alpha = r\,\mathrm{e}^{\mathrm{i}\theta}$ is written in polar coordinates. (i) Argue that the state with the smallest decay rate is independent of θ. (ii) Argue that the stationary state is independent of θ and solves the equation

$$(G(r^2) - \kappa) r P = G(r^2) \partial_r P \tag{6.2}$$

by identifying a probability density and a probability current in Eq.(6.1). (iii) Solve Eq.(6.2) and analyze the solution for a simple mode of saturated gain, for example

$$G(r^2) = G_0 - B r^2 \qquad \text{or} \qquad G(r^2) = \frac{G_0}{1 + b r^2} \tag{6.3}$$

with positive parameters G_0, B, b. Can you find an estimate for the fluctuations in photon number?

Problem 6.2 – From Langevin to Fokker-Planck (5 points)

In the lecture, we have already seen Langevin equations. In the case of classical Brownian motion, they read

$$\dot{p} = -\gamma p + f(t), \qquad \dot{x} = p/m \tag{6.4}$$

where the Langevin force has correlations

$$\langle f(t) f(t') \rangle = 2D\delta(t - t') \tag{6.5}$$

Solve the equation for the momentum formally and calculate the characteristic function

$$\tilde{F}(s, t) = \langle \exp(\mathrm{i}sp(t)) \tag{6.6}$$

for initial conditions $p(0) = p_0$. Show that $\tilde{F}$ solves the equation (no guarantee for signs and numerical factors)

$$\partial_t \tilde{F} = (\gamma s \partial_s - D s^2) \tilde{F} \tag{6.7}$$

It may be needed to take the limit $\gamma \to \infty$. Interpret your result by switching to the Fourier variable conjugate to s.

Remark. The Langevin force $f(t)$ can be understood as a sequence of independent random variables, labelled by the parameter t. More generally, if the random variables x_i have mean value $\bar{x}_i$ and covariance matrix $C_{ij} = \langle x_i x_j \rangle - \bar{x}_i \bar{x}_j$, then the characteristic function is

$$\langle \exp \mathrm{i} \sum_j s_j x_j \rangle = \exp \left[\mathrm{i} \sum_j s_j \bar{x}_j - \frac{1}{2} \sum_{jk} s_j s_k C_{jk} \right] \tag{6.8}$$

This has a natural generalization to integrals $\int \mathrm{d}t \, c(t) x_t$ over a stochastic "process" $x_t = x(t)$.

Problem 6.3 – Fluctuation-dissipation theorem for a two-level atom (8 points)

Calculate for a two-level atom in a thermal state (i.e., $p_e/p_g = \exp(-\hbar\omega_A/k_B T)$) the spectrum of the dipole fluctuations,

$$S(\omega) = |d|^2 \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{-\mathrm{i}\omega t} \langle \sigma^\dagger(t) \sigma(t) \rangle \tag{6.9}$$

where d is the dipole matrix element. (You can perform this calculation directly, without using the quantum regression formula.) Calculate also the linear polarizability $\alpha(\omega)$, i.e.,

$$d \langle \sigma(t) \rangle = \alpha(\omega) E \, \mathrm{e}^{-\mathrm{i}\omega t} \tag{6.10}$$

by applying perturbation theory in the laser field amplitude E to the Bloch equations. (It turns out that $\alpha(\omega)$ is temperature-dependent.) Consider the fluctuation-dissipation theorem (both $\omega > 0$ and $\omega < 0$ are relevant)

$$S(\omega) = \frac{2\hbar}{\mathrm{e}^{\hbar\omega/k_B T} - 1} \mathrm{Im} \, \alpha(\omega) \tag{6.11}$$

and check under which approximations this relation holds.

Remark. The FD-theorem should hold if both sides are calculated to the same precision. This can be a useful check for systematic approximation schemes.