Advanced Quantum Optics

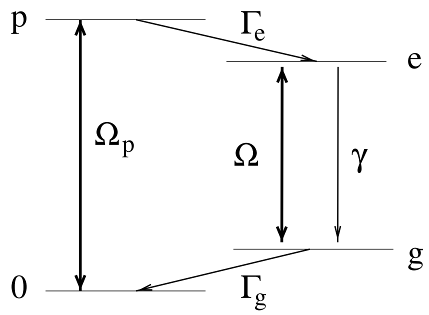
Winter term 2005/06

Carsten Henkel/Martin Wilkens

Exercise session 2 Hand out: 04 November 2005 hand in: 11 November 2005

Problem 2.1 – Inversion and four-level atoms (10 points)

In the lecture, we have introduced a four-level scheme as shown in the figure. The main idea is produce inversion for the 'laser transition' between the states $|e\rangle$ and $|g\rangle$, by pumping of the upper state $|e\rangle$ via the state $|p\rangle$. In this exercise, you compute the 'pumping rate' into $|e\rangle$.



(i) For this atomic level scheme, write the Bloch equations for the density matrix elements $\dot{\rho}_{\alpha\beta}$ with $\alpha, \beta =$ p, e, g, 0, when laser fields are applied on the transitions p $\leftrightarrow$ 0 and e $\leftrightarrow$ g.

(ii) Focus on the density matrix elements for the levels $|p\rangle$ and $|0\rangle$ and consider the limit that the relaxation rate Γ_e is the largest rate in the corresponding Bloch equations. Show that in this limit, the stationary value of the off-diagonal element ("optical coherence") is given by

$$\rho_{p0} = -\frac{\Omega}{2i\Gamma_e}\left(\rho_{pp} - \rho_{00}\right)$$

where $\Delta = \omega_p - \omega_{p0}$ is the detuning of the pumping laser with respect to resonance and $\Omega = -2(\mathbf{d}_{0p}^* \cdot \mathbf{E}_p)/\hbar$ the corresponding Rabi frequency.

(iii) Insert the above expression for ρ_{p0} into the equations for the populations ρ_{pp} and ρ_{00} and consider the regime of 'weak excitation' where $\rho_{00} \approx 1$ is essentially unaffected. Compute to lowest order in the Rabi frequency Ω the population ρ_{pp} and justify why the 'pumping rate' λ_e into $|e\rangle$ is given by

$$\lambda_e = \Gamma_e \rho_{pp} = \frac{|\Omega|^2}{\Gamma_e}.$$

(iv) Give an order of magnitude estimate, assuming that Γ_e is comparable to the free-space radiative decay rate $|\mathbf{d}_{0p}|^2 \omega_{p0}^3 / 3\pi\hbar\varepsilon_0 c^3$.

Problem 2.2 – Quantum theory of mode damping ($5 + \pi^2$ points)

In the lecture, we will see that the dynamics of the field in an empty cavity with losses can be described by the following master equation for the single-mode density operator ρ:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{\mathrm{i}\,\hbar}\left[H_{\mathrm{cav}}, \rho\right] - \frac{\kappa}{2}\left\{a^\dagger a, \rho\right\} + \kappa a \rho a^\dagger$$

where κ is a rate and $\{\cdot, \cdot\}$ is the anticommutator.

(a) Confirm that this equation really describes cavity loss by working out the equation of motion for the mean field amplitude and the mean photon number

$$\langle a \rangle = \mathrm{tr}\,(a\rho), \qquad \bar{n} = \langle a^\dagger a \rangle = \mathrm{tr}\,(a^\dagger a \rho).$$

Give an interpretation of the rate κ.

(b) Construct a reservoir and an interaction Hamiltonian such that one gets the master equation above when the reservoir is traced out in the Born-Markov approximation (weak coupling, short memory time). Follow as closely as possible the derivation of the master equation for the two-level atom given in the last semester.

Hint. Think about $H_{\mathrm{int}} = aB^\dagger + a^\dagger B$ and play with the correlation functions of the operators B, $B^\dagger$. Note that this interaction corresponds to linearly coupled oscillators.

Problem 2.3 – Phase diffusion (10 points)

To characterize the full dynamics of the laser field within the semiclassical framework, one adopts a statistical description in terms of a probability distribution for amplitude and phase of the laser field. We focus here on the dynamics of the phase. We shall see that the phase actually diffuses due to spontaneous emission; this is described by the following equation for the distribution function $P(\phi, t)$

$$\frac{\partial}{\partial t} P = D \frac{\partial^2}{\partial^2 \phi} P$$

It seems reasonable that the phase is restricted to the interval $[0, 2\pi]$ and that P is 2π-periodic.

(a) Taking into account this periodicity, find the solution to this equation with the initial condition $P(\phi, 0) = \delta(\phi - \phi_0)$. This solution, $P(\phi, t|\phi_0)$, is also called the Green function of the diffusion equation.

(b) Convince yourself that the equilibrium solution to the phase diffusion equation is $P^{(ss)}(\phi) = 1/2\pi$.

(c) With these two ingredients, we can enter the calculation of the field's temporal coherence function $\langle E^*(t + \tau) E(t) \rangle$, when the average is taken over the phase fluctuations only (the amplitude is assumed to be fixed). Justify in words the following rule

$$\langle E^*(t+\tau)E(t)\rangle = I_{\rm ss}\int \mathrm{d}\phi\,\mathrm{d}\phi_0\,\mathrm{e}^{-\mathrm{i}\phi}P(\phi,\tau|\phi_0)\mathrm{e}^{\mathrm{i}\phi_0}P^{(ss)}(\phi_0)\mathrm{e}^{\mathrm{i}\omega_L\tau}.$$

(d) Compute the field coherence function according to this rule. You know that its Fourier transform with respect to τ gives the spectrum of the laser field. Confirm that the width of this spectrum is given by the phase diffusion coefficient D. In the lecture, you shall see that $D \propto 1/\langle I\rangle$ so that the spectrum narrows with increasing intensity (Schawlow-Townes formula). Experimentally, this is a key signature to verify that a laser is actually working.