Introduction to Quantum Optics

Winter term 2006/07
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Problem set 2 Hand out: 26 Oct 2006
Hand in: 09 Nov 2006

Problem 2.1 – Photons in a cavity (8 points)

Consider a one-dimensional cavity where the mode functions for the electromagnetic field take the form

$$\mathbf{f}_\kappa(\mathbf{x}) = N\mathbf{e}f(x) \tag{2.1}$$

where N is a normalization factor, x the coordinate along the cavity axis and $\mathbf{e}$ a spatially constant polarization vector.

(1) Find $\mathbf{e}$ such the mode function is transverse.

(2) Assume periodic boundary conditions on the interval $0 \leq x \leq L$ and construct a real-valued $f(x)$ that solves the Helmholtz equation. Find N and an integration domain for the y, z coordinates such that the mode function is normalized as in the lecture.

(3) What is the eigenfrequency ω_κ of this mode? What changes if you adopt boundary conditions for a cavity with perfectly reflecting walls at $x = 0, L$?

(4) Continue with periodic boundary conditions and real-valued modes and bring the total field momentum

$$\mathbf{P} = \varepsilon_0 \int \mathrm{d}^3x\, \mathbf{E} \times \mathbf{B} \tag{2.2}$$

in the form of a sum over mode indices κ and annihilation and creation operators a_κ, $a_\kappa^\dagger$. Use the mode expansion from the lecture for the vector potential to compute the fields. Comment on your result.

Problem 2.2 – Photon dynamics in a single mode (5 points)

We have derived the following wave equation for the vector potential from the Hamiltonian for the particles+field system:

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\partial_t^2 \mathbf{A} = \mu_0 \mathbf{j}_\perp \tag{2.3}$$

where $\mathbf{j}_\perp$ is the transverse current density. In this problem, we focus on a given, time-dependent function $\mathbf{j}_\perp$. Feel free to choose a convenient form if you want to make explicit calculations.

(1) Using the mode expansion for the vector potential in plane waves derived in the lecture, find from Eq.(2.3) the equation of motion for an annihilation operator a_κ (Heisenberg picture).

(2) Solve this equation for a function $\mathbf{j}_\perp$ that oscillates at a frequency near ω_κ and has a temporal envelope (switched on and off after $t = 0$).

(3) Take the average of $a_\kappa(t)$ in the vacuum state of the field and speculate about the meaning of your result.

(4) Sketch qualitatively the average photon number $\langle a_\kappa^\dagger(t) a_\kappa(t) \rangle$ for $t > 0$. (Bonus points.)

Problem 2.3 – Blackbody radiation (7 points)

You have probably learned previously that the electromagnetic energy density u of a 'photon gas' at thermal equilibrium is given by the Planck spectrum:

$$u = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{\hbar(\omega/c)^3}{\pi(\mathrm{e}^{\hbar\omega/k_B T} - 1)} \tag{2.4}$$

(1) Derive this result from QED by working out the expectation value of the Hamiltonian

$$H = \sum_\kappa \hbar\omega_\kappa \left(a_\kappa^\dagger a_\kappa + \tfrac{1}{2} \right)$$

and subtracting the result at zero temperature.

(2) If you start from the *local* energy density

$$u = \frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2$$

using the mode expansions of the electric and magnetic fields, you need some assumptions for expectation values like $\langle a_\kappa^\dagger a_{\kappa'} \rangle$. Identify these and speculate about how to justify them.

(3) With plane wave modes, you finally reach an integral of the form

$$u - u_{\text{vac}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\hbar\omega_k}{\mathrm{e}^{\hbar\omega/k_B T} - 1} \tag{2.5}$$

where $\omega_k = c|\mathbf{k}|$. This leads directly to the Planck formula. (Bonus points.)