Introduction to Quantum Optics

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Carsten Henkel/Martin Wilkens

Problem set 4 Hand out: 23 Nov 2006
Hand in: 07 Dec 2006

Problem 4.1 – Displacement operator and Baker-Hausdorff (4 points)

Prove the Baker-Hausdorff formula for the displacement operator in its normal-ordered form:

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) = \exp(\alpha a^\dagger) \mathrm{e}^{-|\alpha|^2/2} \exp(-\alpha^* a) \tag{4.1}$$

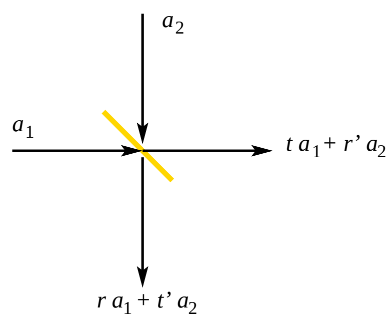
and conclude that the following 'displacement rule' holds

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha \tag{4.2}$$

Problem 4.2 – Beam splitter quantum state transformation (16 points)

A beam splitter, as sketched in Fig.1, mixes two 'input modes' and gives two 'output modes':

$$\begin{pmatrix} a_1(\text{out}) \\ a_2(\text{out}) \end{pmatrix} = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{4.3}$$



(1) Describe the interpretation of the amplitudes r, t, r', t' by assuming that the a_i are complex amplitudes for the incoming waves ('correspondence principle').

We require that if the incoming modes are independent, then the same holds for the outgoing modes, $\left[a_i(\text{out}),\, a_j(\text{out})^\dagger\right] = \delta_{ij}$. (2) Show that this is satisfied if $|t|^2 + |r'|^2 = 1$. Is this relation identical to energy conservation at the beam splitter? (No! Unless $|r| = |r'|$ which often holds because of symmetry arguments like time reversal and parity.)

Specialize to the case $t = t' = \cos\theta$, $r = -r' = \sin\theta$ and (3) show that the following unitary transformation U implements the mapping from incoming to outgoing quantum states in the following sense:

$$a_1(\text{out}) = U^\dagger a_1 U = a_1 \cos\theta - a_2 \sin\theta \tag{4.4}$$

with

$$U = \exp\left[\theta \left(a_2^\dagger a_1 - a_1^\dagger a_2\right)\right] \tag{4.5}$$

(4) Compute the outgoing state $|\psi_{\rm out}\rangle = U|\psi_{\rm in}\rangle$ for an incoming two-photon state $|\psi_{\rm in}\rangle = a_1^\dagger a_2^\dagger|{\rm vac}\rangle = |1,1\rangle$ with exactly one photon in each mode. Discuss the behaviour with the 'mixing angle' θ and check whether one can produce a entangled two-photon state $|\psi_{\rm out}\rangle \propto |2,0\rangle + |0,2\rangle$. Describe in words what is different here compared to incoming coherent states.