Carsten Henkel

Introduction to Quantum Optics

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The preliminary programme for this lecture:

1. Field quantization: QED
   - — canonical scheme in the Coulomb gauge
   - — quantum (or vacuum) fluctuations
   - — field states (photon number states, thermal, coherent, squeezed states)
2. Atom-field dynamics
   - — atom-light coupling
   - — the model of a two-level atom
   - — spontaneous emission
   - — elementary photodetector theory
3. Applications in photonics
   - — quantum laser theory
   - — quantum state manipulation with optical elements

Chapter 1

The quantized field

We give an elementary introduction to the quantization of the electromagnetic field. We adopt the Coulomb gauge and a simple canonical framework. We then discuss examples for the quantum states of the field and the properties of vacuum fluctuations.

1.1 Canonical quantization

1.1.1 Fields

Maxwell equations. We want to quantize the vacuum Maxwell equations for the electromagnetic field. With a given charge and current density, these read

$$\begin{array}{ll} \nabla\cdot\mathbf{B}=0 & \varepsilon_0\nabla\cdot\mathbf{E}=\rho \\ \nabla\times\mathbf{E}+\partial_t\mathbf{B}=0 & \nabla\times\mathbf{B}-\frac{1}{c^2}\partial_t\mathbf{E}=\mu_0\mathbf{j} \end{array} \tag{1.1}$$

Introducing the scalar and vector potentials via

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{1.2}$$

$$\mathbf{E} = -\nabla\phi-\partial_t\mathbf{A}, \tag{1.3}$$

the left column of (1.1) is identically fulfilled. The Coulomb law then becomes

$$-\varepsilon_0\triangle\phi-\varepsilon_0\nabla\partial_t\mathbf{A}=\rho \tag{1.4}$$

If we impose the Coulomb gauge, $\nabla\cdot\mathbf{A}=0$, the vector potential drops out and the scalar potential is determined by the charge density alone:

$$-\varepsilon_0\triangle\phi=\rho \tag{1.5}$$

In this gauge, the scalar potential is thus not a dynamical degree of freedom of the field: its dynamics is 'enslaved' by that of the charges. This holds with suitable boundary conditions such that the homogeneous equation $\triangle\phi=0$ has no nontrivial solutions. In free space, with $\phi(\mathbf{x}\to\infty)\to 0$, we get

$$\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\mathrm{d}^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{1.6}$$

which is a superposition of well-known Coulomb potentials.

Wave equation. We are left with the wave equation for the vector potential

$$\nabla\times(\nabla\times\mathbf{A})+\frac{1}{c^2}\partial_t^2\mathbf{A}=\mu_0\left(\mathbf{j}-\varepsilon_0\partial_t\nabla\phi\right)\equiv\mu_0\mathbf{j}_\perp. \tag{1.7}$$

On the right hand side, we have introduced the 'transverse current'. Its divergence is zero because of Eq.(1.5) and charge conservation:

$$\nabla\cdot\mathbf{j}_\perp=\nabla\cdot\mathbf{j}+\partial_t\rho=0. \tag{1.8}$$

The name 'transverse' comes from the fact that in spatial Fourier components, the current $\mathbf{j}_\perp(\mathbf{k})$ must be perpendicular to $\mathbf{k}$. One also says that the vector potential in the Coulomb gauge is transverse because $\nabla\cdot\mathbf{A}=0$. The transversality of the source term in (1.7) ensures that if $\mathbf{A}$ is transverse at one time, it is also transverse at all later times.

Conservation laws. Charge: $\partial_t\rho+\nabla\cdot\mathbf{j}=0$.
Energy (Poynting theorem):

$$\partial_t u+\nabla\cdot\mathbf{S}=-\mathbf{j}\cdot\mathbf{E} \tag{1.9}$$

$$u=\frac{\varepsilon_0}{2}\mathbf{E}^2+\frac{1}{2\mu_0}\mathbf{B}^2 \qquad \text{energy density} \tag{1.10}$$

$$\mathbf{S}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \qquad \text{Poynting vector} \tag{1.11}$$

Momentum (see exercise).

1.1.2 Matter

Let us consider that all matter is made from charged point particles with charges e_α and positions $\mathbf{r}_\alpha$. The electric charge and current densities are then given by

$$\rho(\mathbf{x},t) = \sum_\alpha e_\alpha \delta(\mathbf{x} - \mathbf{r}_\alpha(t)), \qquad \mathbf{j}(\mathbf{x},t) = \sum_\alpha e_\alpha \mathbf{v}_\alpha(t)\delta(\mathbf{x} - \mathbf{r}_\alpha(t)) \quad (1.12)$$

The sum runs over all the particles. Charge conservation is ensured provided that $\dot{\mathbf{r}}_\alpha = \mathbf{v}_\alpha$.

The point charges are thus the 'sources' for the electromagnetic field. But their motion is also influenced by the fields via the Newton-Lorentz equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_\alpha \mathbf{v}_\alpha}{\sqrt{1 - v_\alpha^2/c^2}} = e_\alpha \left(\mathbf{E}(\mathbf{r}_\alpha) + \mathbf{v}_\alpha \times \mathbf{B}(\mathbf{r}_\alpha)\right) \quad (1.13)$$

We use a relativistic framework here, and with the Lorentz factor, the time derivative is actually the one for the relativistic particle momentum. Note that in these equations, the coordinate $\mathbf{r}_\alpha$ enters generally in a nonlinear way. This differs from the Maxwell equations that are linear in the fields and potentials and whose solutions are linear in the charge and current distributions. (The superposition principle can be applied, see Eq.(1.6).) All nonlinear effects in optics can ultimately be traced back to the nonlinear response of matter to an applied electromagnetic field.

Quantization. The theory outlined so far describes physics at the end of the 19th century. It is unable to describe a stable state of matter because of the 'radiation catastrophe': positive and negative charges circle around each other, radiate electromagnetic waves and lose energy. Stable bound states exist only when the particle dynamics is quantized—recall the hydrogen atom.

Let us recall the typical energy and length scales for the hydrogen atom. The energy levels in Hydrogen are given by

$$E_n = -\frac{\mathrm{Ryd}}{n^2} = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2} \quad (1.14)$$

where the Rydberg constant is $1\,\mathrm{Ryd} \approx 13.6\,\mathrm{eV}$. (In cgs units, drop the factor $(4\pi\varepsilon_0)^2$.) The size of the Hydrogen atom is of the order of the Bohr

radius

$$a_0 = \frac{4\pi\varepsilon_0\,\hbar^2}{me^2} \quad (1.15)$$

The typical wavelength of an electromagnetic wave resonant with a transition in Hydrogen is therefore of the order of

$$\lambda = \frac{\hbar c}{\mathrm{Ryd}} = \frac{2}{\alpha_{\mathrm{fs}}}a_0 \quad (1.16)$$

$$\frac{1}{\alpha_{\mathrm{fs}}} = \frac{4\pi\varepsilon_0\,\hbar c}{e^2} \approx 137 \quad (1.17)$$

Here, α_{fs} is the fine structure constant. Its inverse can be understood as a measure of the speed of light in 'atomic units' (the natural units for the Hydrogen problem). The value $1/\alpha_{\mathrm{fs}} \approx 137$ is fairly large. This means two things:

— the size of the hydrogen atom is small compared to the wavelength of resonant radiation: the Lorentz force (1.13) has therefore only a weak dependence on $\mathbf{r}_\alpha$;

— the typical velocity of an electron in the Hydrogen atom is in the non-relativistic regime: we can therefore use non-relativistic mechanics to describe the matter response.

This will justify several approximations for the atom-light interaction that we are going to make in the rest of the lecture.

1.1.3 Lagrange-Hamilton formulation

Lagrangian. We now proceed to quantize the wave equation (1.7) in the 'canonical way'. The first step is to guess the corresponding Lagrangian. A good guess is the Lagrangian

$$L = -\sum_\alpha m_\alpha c^2 \sqrt{1 - \dot{\mathbf{r}}_\alpha^2/c^2} + \int \mathrm{d}^3x\,\mathcal{L}_{F+I} \quad (1.18)$$

where the Lagrangian density for the field and its interaction with the electric charges is given by

$$\mathcal{L}_{F+I} = \frac{\varepsilon_0}{2}\left(\dot{\mathbf{A}} + \nabla\phi\right)^2 - \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 - \phi\rho + \mathbf{A}\cdot\mathbf{j}. \quad (1.19)$$

Via the Euler-Lagrange equations, one gets the Maxwell equations (1.5,1.7) for the fields and the Newton-Lorentz equation (1.13) for the particles. Note that the Lagrangian (1.18,1.19) is invariant under gauge transformations

$$\phi \mapsto \phi - \partial_t \chi, \qquad \mathbf{A} \mapsto \mathbf{A} + \nabla \chi \tag{1.20}$$

where $\chi(\mathbf{x}, t)$ is an arbitrary smooth function. This gauge invariance is connected to charge conservation.

Coulomb gauge. We now proceed to specialize to the Coulomb gauge, simplify the Lagrangian and derive the Hamiltonian. We start with the terms involving the scalar potential in the Lagrangian (1.19). The mixed term is

$$\dot{\mathbf{A}} \cdot \nabla \phi = \nabla \cdot (\dot{\mathbf{A}} \phi) - \phi \nabla \cdot \dot{\mathbf{A}} \tag{1.21}$$

The first term is a divergence, and leads to a surface integral when integrated over the volume. We adopt the usual boundary condition that at the (infinitely remote) surface, the fields vanish: then this term is zero. The second term is zero in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$.

The term quadratic in the scalar potential is

$$(\nabla \phi)^2 = \nabla \cdot (\phi \nabla \phi) - \phi \triangle \phi = \nabla \cdot (\phi \nabla \phi) + \phi \rho / \varepsilon_0 \tag{1.22}$$

using the Laplace equation (1.5). The second term thus combines with the interaction part $-\phi\rho$ in the Lagrangian that becomes $-\frac{1}{2}\phi\rho$. This energy can be interpreted as the Coulomb interaction energy between the charges:

$$\begin{aligned} V_{\text{Coul}} &= \frac{1}{2} \int \mathrm{d}^x \, \phi \rho = \frac{1}{2} \int \mathrm{d}^3 x \, \mathrm{d}^3 x' \, \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{4\pi\varepsilon_0 \, |\mathbf{x} - \mathbf{x}'|} && (1.23) \\ &= \frac{1}{2} \sum_{\alpha\beta} \frac{e_\alpha e_\beta}{4\pi\varepsilon_0 \, |\mathbf{r}_\alpha - \mathbf{r}_\beta|} && (1.24) \end{aligned}$$

where the factor $\frac{1}{2}$ ensures that all pairs of charges are only counted once. The divergent self-interaction for $\mathbf{r}_\alpha = \mathbf{r}_\beta$ also appears here. It is usually discarded. *The key point to note is that in the Coulomb gauge, the contribution of the scalar potential depends only on the particle coordinates. It is not a proper degree of freedom of the fields.*

To summarize, in the Coulomb gauge, the Lagrangian can be split into the following form:

$$L = -\sum_\alpha m_\alpha c^2 \sqrt{1 - v_\alpha^2 / c^2} - V_{\text{Coul}}(\{\mathbf{r}_\alpha\}) + \int \mathrm{d}^3 x \, \mathcal{L}_{\text{F+I}}^{\perp} \tag{1.25}$$

with

$$\mathcal{L}_{\text{F+I}}^{\perp} = \frac{\varepsilon_0}{2} \dot{\mathbf{A}}^2 - \frac{1}{2\mu_0} (\nabla \times \mathbf{A})^2 + \mathbf{A} \cdot \mathbf{j}. \tag{1.26}$$

We have added the subscript $_\perp$ to remind ourselves that this is only valid if the vector potential is transverse.

Hamiltonian. For the Hamiltonian, we need the canonical momenta conjugate to $\mathbf{r}_\alpha$ and $\mathbf{A}$:

$$\begin{aligned} \mathbf{p}_\alpha &= \frac{\partial L}{\partial \dot{\mathbf{r}}_\alpha} = \frac{m \dot{\mathbf{r}}_\alpha}{\sqrt{1 - \dot{\mathbf{r}}_\alpha^2 / c^2}} + e_\alpha \mathbf{A}(\mathbf{r}_\alpha) && (1.27) \\ \mathbf{\Pi}(\mathbf{x}) &= \frac{\delta L}{\delta \dot{\mathbf{A}}(\mathbf{x})} = \varepsilon_0 \dot{\mathbf{A}}(\mathbf{x}) && (1.28) \end{aligned}$$

The particle momentum contains the relativistic kinetic momentum and an electromagnetic contribution. For the field momentum, it looks as if ε_0 where the 'mass' and $\dot{\mathbf{A}}$ the velocity.

Functional derivative. In the field case, we have a continuous collection of degrees of freedom, labelled by the space-points $\mathbf{x}$. This requires a generalization of the notion of a derivative to the infinite-dimensional case: the 'functional derivative' $\delta L / \delta \dot{\mathbf{A}}(\mathbf{x})$. It is the generalization of a gradient.

In mathematical terms, if we have a 'functional' $L[\mathbf{A}(\mathbf{x})]$, i.e. a mapping from the space of vector fields into the real numbers, its functional derivative is defined by the following approximation:

$$\begin{aligned} L[\mathbf{A}(\mathbf{x}) + \delta \mathbf{A}(\mathbf{x})] &\approx L[\mathbf{A}(\mathbf{x})] \\ &+ \int \mathrm{d}^3 x \, \left. \frac{\delta L}{\delta \mathbf{A}(\mathbf{x})} \right|_{\mathbf{A}(\mathbf{x})} \delta \mathbf{A}(\mathbf{x}) + \mathcal{O}(\delta \mathbf{A}^2) && (1.29) \end{aligned}$$

Here, the second line is an example of a linear functional because the integral is linear in $\delta \mathbf{A}(\mathbf{x})$. The function with which the small deviation $\delta \mathbf{A}(\mathbf{x})$

is weighted under the integral *defines* the functional derivative. A mathematical theorem ensures that in a suitable space of functions, all linear functionals take this integral form.

It is a simple exercise to derive with (1.29) expression (1.28) for the field momentum $\mathbf{\Pi}$. We shall return to a less trivial example below.

The Hamiltonian is given by

$$H = \sum_\alpha \mathbf{p}_\alpha \cdot \dot{\mathbf{r}}_\alpha + \int \mathrm{d}^3x\, \mathbf{\Pi} \cdot \dot{\mathbf{A}} - L \tag{1.30}$$

where the field part is the obvious generalization to a continuous set of degrees of freedom. Putting everything together, we get

$$\begin{aligned} H &= \sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4} + V_{\mathrm{Coul}}(\{\mathbf{r}_\alpha\}) \\ &\quad + \int \mathrm{d}^3x \left[\frac{\mathbf{\Pi}^2}{2\varepsilon_0} + \frac{(\nabla \times \mathbf{A})^2}{2\mu_0}\right] \end{aligned} \tag{1.31}$$

where the interaction between matter and (transverse) field arises due to the 'minimal coupling' prescription only. (The term linear in $\mathbf{j}$ in the Lagrangian cancels with the term linear in $\dot{\mathbf{r}}_\alpha$ in (1.30).)

Canonical equations. The motion of particles and fields in the Hamiltonian formalism can be described in a compact way in terms of Poisson brackets. We discuss this in some detail because they provide another example of functional derivatives and because they bear strong similarities to the commutators of the quantum theory. In addition, it turns out to be tricky to get the transverse wave equation (1.7).

The Poisson bracket provides the time evolution of any function Q (or functional) of the coordinates and momenta by

$$\dot{Q} = \{H, Q\} \tag{1.32}$$

where we define

$$\begin{aligned} \{A, B\} &= \sum_\alpha \frac{\partial A}{\partial \mathbf{p}_\alpha} \cdot \frac{\partial B}{\partial \mathbf{r}_\alpha} - \frac{\partial A}{\partial \mathbf{r}_\alpha} \cdot \frac{\partial B}{\partial \mathbf{p}_\alpha} \\ &\quad + \int \mathrm{d}^3x \left[\frac{\delta A}{\delta \mathbf{\Pi}(\mathbf{x})} \cdot \frac{\delta B}{\delta \mathbf{A}(\mathbf{x})} - \frac{\delta A}{\delta \mathbf{A}(\mathbf{x})} \cdot \frac{\delta B}{\delta \mathbf{\Pi}(\mathbf{x})}\right] \end{aligned} \tag{1.33}$$

Here, functional derivatives with respect to $\mathbf{A}$ and $\mathbf{\Pi}$ appear for the fields. The Poisson bracket is antisymmetric in A and B and satisfies a 'Jacobi identity' (as does the commutator).

By working out $\{H, \mathbf{r}\}$, only the second term of the first line in (1.33) contributes, and we get (after some calculations) the relativistic relation between velocity and momentum, Eq.(1.27). This is left as an exercise. Similarly, one gets $\dot{\mathbf{A}} = \mathbf{\Pi}/\varepsilon_0$.

A more complicated calculation is needed for $\dot{\mathbf{\Pi}}$ where we have to evaluate

$$-\frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4} \tag{1.34}$$

We note first that

$$\frac{\delta \mathbf{A}(\mathbf{r}_\alpha)}{\delta \mathbf{A}(\mathbf{x})} = \delta(\mathbf{r}_\alpha - \mathbf{x}) \tag{1.35}$$

because the 'evaluation functional' $\mathbf{A} \mapsto \mathbf{A}(\mathbf{r}_\alpha)$ is of course a linear functional. (For mathematicians, this property *defines* the δ-function.) To proceed, we use the usual rules of differential calculus and get

$$-\frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4} = \delta(\mathbf{r}_\alpha - \mathbf{x}) \frac{e_\alpha (\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)) c^2}{\sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4}} \tag{1.36}$$

Performing the same calculations as for the particles' equations of motion, this can be written as the current density $\mathbf{j}(\mathbf{x})$.

The last term now involves the derivative

$$-\frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \frac{1}{2\mu_0} \int \mathrm{d}^3x (\nabla \times \mathbf{A})^2 \tag{1.37}$$

that we handle with the mathematical definition (1.29). Consider a small change $\mathbf{a}(\mathbf{x})$ of the vector potential. To linear order, this gives a change

$$\begin{aligned} &(\nabla \times (\mathbf{A} + \mathbf{a}))^2 - (\nabla \times \mathbf{A})^2 \approx 2(\nabla \times \mathbf{a}) \cdot (\nabla \times \mathbf{A}) \\ &= 2\nabla \cdot [\mathbf{a} \times (\nabla \times \mathbf{A})] + 2\mathbf{a} \cdot [\nabla \times (\nabla \times \mathbf{A})] \end{aligned} \tag{1.38}$$

the first term is a divergence and vanishes after integrating over all space. The second one contains $\mathbf{a}$ multiplied with a weighting function so that we get

$$-\frac{1}{2\mu_0} \frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \int \mathrm{d}^3x (\nabla \times \mathbf{A})^2 = -\frac{1}{\mu_0} \nabla \times (\nabla \times \mathbf{A}) \tag{1.39}$$

Putting everything together, we have for the equation of motion of the vector potential:

$$\dot{\mathbf{\Pi}} = \varepsilon_0 \ddot{\mathbf{A}} = \mathbf{j}(\mathbf{x}) - \frac{1}{\mu_0}\nabla \times (\nabla \times \mathbf{A}) \qquad \ldots\text{wrong} \tag{1.40}$$

which is nearly equivalent to the wave equation (1.7). The point is that the source term is the 'full current', not its transverse part. This is actually an error in our calculation because we did not take into account the fact that the vector potential is restricted to be transverse.

A simple way to repair this is to use $\mathbf{A}_\perp$ in the Lagrangian (1.25) and the Hamiltonian (1.31). Now, the link between $\mathbf{A}_\perp$ and the 'full' $\mathbf{A}$ is a linear functional (actually, a linear projector). This can be seen in the following way: consider an arbitrary $\mathbf{A}$ and perform a gauge transformation (1.20) to remove the nonzero divergence. This fixes the 'gauge function' to satisfy

$$\triangle\chi = -\nabla\cdot\mathbf{A} \tag{1.41}$$

whose solution (vanishing at infinity) is given by a 'superposition of Coulomb potentials':

$$\chi(\mathbf{x}) = \frac{1}{4\pi}\int \mathrm{d}^3x'\, \frac{\nabla'\cdot\mathbf{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{1.42}$$

where ∇' means the gradient with respect to $\mathbf{x}'$. After the gauge transformation, the now transverse vector potential is given by

$$\mathbf{A}_\perp(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \frac{1}{4\pi}\int \mathrm{d}^3x'\,[\nabla'\cdot\mathbf{A}(\mathbf{x}')]\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{1.43}$$

Now, $\nabla(1/r) = -\nabla'(1/r)$, and after one integration by parts we have

$$A_{\perp i}(\mathbf{x}) = \int \mathrm{d}^3x'\, \delta^\perp_{ij}(\mathbf{x}-\mathbf{x}')A_j(\mathbf{x}) \tag{1.44}$$

where the so-called 'transverse δ-function' is given by

$$\delta^\perp_{ij}(\mathbf{x}-\mathbf{x}') = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{4\pi}\partial'_j\partial'_i\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{1.45}$$

A careful evaluation of the second derivatives yields the explicit result

$$\delta^\perp_{ij}(\mathbf{x}-\mathbf{x}') = \frac{2}{3}\delta_{ij}\delta(\mathbf{x}-\mathbf{x}') - \frac{1}{4\pi}\left(\frac{\delta_{ij}}{r^3} - 3\frac{r_i r_j}{r^5}\right), \qquad \mathbf{r} = \mathbf{x}-\mathbf{x}' \tag{1.46}$$

By construction, the transverse δ-function acts like a usual δ-function on vector fields that are already transverse. We can interpret it as the 'unit operator' in the space of transverse vector fields.

Finally, if we write $\mathbf{A}_\perp$ in the Hamiltonian, the equations of motion for $A_{\perp i}(\mathbf{x})$ lead to the following term

$$\int \mathrm{d}^3x' \frac{\delta H}{\delta\Pi_j(\mathbf{x}')}\frac{\delta A_{\perp i}(\mathbf{x})}{\delta A_j(\mathbf{x}')} = \int \mathrm{d}^3x' \frac{\delta H}{\delta\Pi_j(\mathbf{x}')}\delta^\perp_{ij}(\mathbf{x}-\mathbf{x}') = \left(\frac{\delta H}{\delta\mathbf{\Pi}(\mathbf{x}')}\right)_{\perp i} \tag{1.47}$$

The functional derivative with respect to $\mathbf{A}$ makes the transverse δ-function appear. The integral over $\mathbf{x}'$ then projects the functional derivative with respect to $\mathbf{\Pi}$ into the transverse subspace. In this way, the time derivative of $\mathbf{A}_\perp$ is transverse, as it should do in order to maintain the Coulomb gauge at all times.

The same procedure applies to the equation of motion for the field momentum: the bottomline is that the source current $\mathbf{j}$ is 'transversalized', leading to the correct wave equation (1.7).

1.1.4 Quantization

Mode expansion. The next step is to look for 'normal modes' of this field theory. We first identify some general requirements for the modes. The passage to plane waves is a little bit tricky and is treated in detail in the exercises. Note that we ignore for the moment the matter-field coupling: wo focus on the field Hamiltonian only.

We adopt the expansion

$$\begin{pmatrix} \mathbf{A}(\mathbf{x},t) \\ \mathbf{\Pi}(\mathbf{x},t) \end{pmatrix} = \sum_\kappa \mathbf{f}_\kappa(\mathbf{x}) \begin{pmatrix} q_\kappa(t) \\ p_\kappa(t) \end{pmatrix} \tag{1.48}$$

where the 'mode functions' $\mathbf{f}_\kappa(\mathbf{x})$ carry the space dependence and the 'coordinates' $q_\kappa(t)$ and 'momenta' $p_\kappa(t)$ the time-dependence.

The Coulomb gauge requires, of course, $\nabla\cdot\mathbf{f}_\kappa(\mathbf{x}) = 0$: the mode functions must be transverse.

We want the Hamiltonian to adopt a simple form using these modes. The space integral over $\mathbf{\Pi}^2$ becomes simple if we impose the modes to be orthogonal:

$$\int \mathrm{d}^3x\, \mathbf{f}_\kappa(\mathbf{x})\cdot\mathbf{f}_{\kappa'}(\mathbf{x}) = N_\kappa\delta_{\kappa\kappa'} \tag{1.49}$$

where N_κ is a normalization constant that we fix later. The momentum-part of the Hamiltonian then becomes

$$\frac{1}{2\varepsilon_0}\int \mathrm{d}^3x\, \mathbf{\Pi}^2 = \sum_\kappa N_\kappa \frac{p_\kappa^2}{2\varepsilon_0} \tag{1.50}$$

Similarly, for the integral over $(\nabla \times \mathbf{A})^2$. Integrating by parts:

$$\begin{aligned} &\int \mathrm{d}^3x\, (\nabla \times \mathbf{f}_\kappa) \cdot (\nabla \times \mathbf{f}_{\kappa'}) \\ &= \int \mathbf{dA} \cdot [\mathbf{f}_\kappa \times (\nabla \times \mathbf{f}_{\kappa'})] + \int \mathrm{d}^3x\, \mathbf{f}_\kappa \cdot [\nabla \times (\nabla \times \mathbf{f}_{\kappa'})] \end{aligned} \tag{1.51}$$

The volume integral, strictly speaking, must be evaluated over a finite volume only, otherwise, we could not work with a discrete set of mode labels κ. The boundary term can nevertheless be made to vanish if either (i) the mode function $\mathbf{f}_\kappa$ or its curl $(\nabla \times \mathbf{f}_{\kappa'})$ is required to vanish on the boundary of the volume or (ii) periodic boundary conditions on 'opposite faces' of a cubic volume are assumed. The case (i) is appropriate for modes in a cavity with perfectly conducting boundaries: then, $\mathbf{f}_\kappa$ is proportional to the electric field, and the integrand in Eq.(1.51) vanishes because the field is normal to the boundary. The case (ii) is the favorite one for theorists because the mode functions can be taken as plane waves. Note that the eigenfrequencies of the two cavities are not the same. We ignore for the moment the complications of complex mode functions (see details below and the exercises) and continue.

The volume integral in (1.51) is reduced to the orthogonality relation if we require the mode functions to be eigenfunctions of the (vector) Helmholtz equation:

$$\nabla \times (\nabla \times \mathbf{f}_\kappa) = \varepsilon_0 \mu_0 \omega_\kappa^2 \mathbf{f}_\kappa \tag{1.52}$$

(This equation is actually equivalent to the scalar Helmholtz equation for all components of $\mathbf{f}_\kappa$ because of transversality.)

With all these assumptions taken together, the Hamiltonian for the field takes the form

$$H_{\mathrm{F}} = \sum_\kappa \left[\frac{N_\kappa}{2\varepsilon_0} p_\kappa^2 + \frac{\varepsilon_0}{2} N_\kappa \omega_\kappa^2 q_\kappa^2 \right] \tag{1.53}$$

We now fix the normalization to be $N_\kappa = 1$ and get a sum of harmonic oscillator Hamiltonians, one for each mode κ with 'mass' ε_0 and frequency ω_κ.

Note: the construction of field modes is a 'classical problem' of electrodynamics, it has nothing to do with quantum mechanics. The word 'quantization volume' that is sometimes used (to ensure that the mode index κ is discrete) is therefore misleading. Quantization is something different, as we shall see now.

Mode operators. Quantization proceeds by promoting the p_κ and q_κ to operators with the commutation relations

$$\frac{\mathrm{i}}{\hbar}[p_\kappa,\, q_{\kappa'}] = \delta_{\kappa\kappa'} \tag{1.54}$$

The choice for this commutator is similar to the one for the particle coordinates and momenta in ordinary quantum mechanics. So in the end, field quantization is nothing else but ordinary quantization, once the dynamics of the field is reduced to a discrete set of 'normal modes'. The procedure that we have followed was first laid out by Dirac. It is called 'canonical quantization'.

The commutator between the fields becomes

$$\frac{\mathrm{i}}{\hbar}[\Pi_i(\mathbf{x}),\, A_j(\mathbf{x}')] = \sum_\kappa f_{\kappa i}(\mathbf{x}) f_{\kappa j}(\mathbf{x}') \stackrel{!}{=} \delta_{ij}^{\perp}(\mathbf{x} - \mathbf{x}') \tag{1.55}$$

The last equality is obtained by applying the canonical quantization scheme directly to the fields, care being taken that the fields (operators) 'live' in the space of transverse vector fields (operators). It means that the mode functions $\mathbf{f}(\mathbf{x})$ form a *complete set* of functions in the transverse field space. To implement this equality, one takes in practice the limit of an infinitely large quantization volume where the sum degenerates into an integral. With discrete functions, one can actually represent only a 'finite volume version' of the transverse δ-function—the one that comes by applying the finite volume boundary conditions to the equation (1.41).

For the harmonic oscillator, creation and annihilation operators are a convenient tool to construct the Hilbert space of quantum states. In our context, these operators, $a_\kappa^\dagger$ and a_κ, correspond to the 'creation' and 'destruction' of one 'photon'. The mode coordinate and momentum operators are given by

$$q_\kappa = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa}}\left(a_\kappa + a_\kappa^\dagger\right) \tag{1.56}$$

$$p_\kappa = \sqrt{\frac{\hbar\varepsilon_0\omega_\kappa}{2}}\mathrm{i}\left(a_\kappa - a_\kappa^\dagger\right) \tag{1.57}$$

where we continue to write ε_0 for the oscillator mass and where the commutation relation is

$$\left[a_\kappa,\, a_{\kappa'}^\dagger\right] = \delta_{\kappa\kappa'} \tag{1.58}$$

The field Hamiltonian then takes the form

$$H_\mathrm{F} = \sum_\kappa \frac{\hbar\omega_\kappa}{2}\left(a_\kappa a_\kappa^\dagger + a_\kappa^\dagger a_\kappa\right) = \sum_\kappa \hbar\omega_\kappa\left(a_\kappa^\dagger a_\kappa + \tfrac{1}{2}\right) \tag{1.59}$$

The last way of writing makes two essential things explicit:

— the energies (the energy eigenvalues!) of a given field mode are quantized in units of $\hbar\omega_\kappa$ (a 'photon energy') and the 'number of photons' is represented by the operator $a_\kappa^\dagger a_\kappa$.

— The ground state of the field corresponds to the state $|\mathrm{vac}\rangle$ such that $a_\kappa|\mathrm{vac}\rangle = 0$ for all κ. This is an energy eigenstate whose energy is infinity, $\frac{1}{2}\sum_\kappa \hbar\omega_\kappa$, the sum over the 'zero-point energies' of all the modes.

To summarize, we give the mode expansions of the vector potential that we have found

$$\mathbf{A}(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa}}\mathbf{f}_\kappa(\mathbf{x})\left(a_\kappa \mathrm{e}^{-\mathrm{i}\omega_\kappa t} + a_\kappa^\dagger \mathrm{e}^{\mathrm{i}\omega_\kappa t}\right). \tag{1.60}$$

We have used here the Heisenberg picture for the vector potential operator. From the Hamiltonian (1.59), it is easy to show that the operator $a_\kappa(t)$ evolves with a complex exponential $\mathrm{e}^{-\mathrm{i}\omega_\kappa t}$. This is also called a 'positive frequency operator'. It can be shown that operators that destroy particles and lower the energy of a quantum state are always positive frequency operators.

Plane wave expansion. For completeness, we give here the plane-wave expansion for the field mode functions. These are complex, and therefore they are normalized according to

$$\int \mathrm{d}^3x\, \mathbf{f}_\kappa^*(\mathbf{x}) \cdot \mathbf{f}_{\kappa'}(\mathbf{x}) = \delta_{\kappa\kappa'} \tag{1.61}$$

instead of Eq.(1.49). This can be ensured with the choice

$$\mathbf{f}_\kappa(\mathbf{x}) = \frac{1}{\sqrt{V}}\mathbf{u}_\kappa \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{1.62}$$

where V is the volume of the box with periodic boundary conditions, $\mathbf{k}$ is a discrete wave vector: it increases in steps of $2\pi/V^{1/3}$ for a cubic box. And $\mathbf{u}_\kappa$ is a 'transverse' polarization vector with the property $\mathbf{k}\cdot\mathbf{u}_\kappa = 0$. There are two mutually orthogonal choices of polarization for a given $\mathbf{k}$. (These can be complex, describing circular polarization.) The frequency of this mode is $\omega_\kappa = |\mathbf{k}|/\sqrt{\varepsilon_0\mu_0} = c|\mathbf{k}|$, as can be seen from the Helmholtz equation (1.52). The magnetic field is oriented along $\mathbf{k}\times\mathbf{u}_\kappa \equiv (\omega_\kappa/c)\mathbf{v}_\kappa$.

Finally, the quantized vector potential, electric and magnetic fields are given in terms of the following plane wave expansion

$$\mathbf{A}_\perp(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa V}}\left(\mathbf{u}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_\kappa t)}a_\kappa + \mathrm{h.c.}\right). \tag{1.63}$$

$$\mathbf{E}_\perp(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar\omega_\kappa}{2\varepsilon_0 V}}\left(\mathrm{i}\mathbf{u}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_\kappa t)}a_\kappa + \mathrm{h.c.}\right). \tag{1.64}$$

$$\mathbf{B}(\mathbf{x},t) = \sum_\kappa \sqrt{\frac{\hbar\omega_\kappa\mu_0}{2V}}\left(\mathrm{i}\mathbf{v}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega_\kappa t)}a_\kappa + \mathrm{h.c.}\right). \tag{1.65}$$

Sometimes, you may encounter these formulas without the factor i in front (and with '$+\mathrm{h.c.}$'). Then the operator $\mathrm{i}a_{\mathbf{k}\lambda}$ is being used instead. Note that Eq.(1.64) gives only the 'transverse' part of the electric field. The 'longitudinal' part, $-\nabla\phi$, is determined according to (1.5) by the charge density.

Note: A useful shortcut to derive the prefactors is the following: for each mode, match the energy density $\frac{1}{2}\varepsilon_0\mathbf{E}^2 + (1/2\mu_0)\mathbf{B}^2$ to the photon energy per quantization volume, $(\hbar\omega_\kappa/V)(a_\kappa^\dagger a_\kappa + \frac{1}{2})$. In free space, the electric and magnetic energy densities are equal. Average over the spatial oscillations in $\mathbf{E}^2$ for simplicity.

Exercise. Show that the momentum of the field can be written as a sum over modes as well. With plane wave modes, the momentum per mode is quantized in units of $\hbar\mathbf{k}$, as expected. Arbitrary cavity modes involving sin or cos functions are not eigenfunctions of the momentum operators, therefore their momentum is not well-defined. A 'cavity photon' therefore does not have a well-defined momentum.

Exercise. Write the equation of motion for a mode operator a_κ and include the source current. Solve it for known time-dependence of current.

1.1.5 Plane wave expansion

In many cases, plane waves are a more natural choice for the mode functions. But they are complex, and this leads to some technical difficulties. We outline below a quantization procedure that directly starts from plane waves.

Plane waves are an obvious choice in a box of volume V with periodic boundary conditions. We can expand

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \tag{1.66}$$

where the plane waves are normalized to unity in the ('quantization') volume V. Some simple manipulations then lead to the Lagrangian

$$L = \frac{\varepsilon_0}{2} \sum_{\mathbf{k}} \left(\dot{\mathbf{A}}_{\mathbf{k}} \cdot \dot{\mathbf{A}}_{-\mathbf{k}} - c^2 k^2 \, \mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{-\mathbf{k}} + \mathbf{A}_{\mathbf{k}} \cdot \mathbf{j}_{-\mathbf{k}} \right). \tag{1.67}$$

This already decomposes into a sum over the wave vectors $\mathbf{k}$. Only waves with $\mathbf{k}$ and $-\mathbf{k}$ are coupled. We can also decompose the complex vector in a basis of orthogonal polarization vectors (there are two of them with $\mathbf{k} \perp \mathbf{u}_{\mathbf{k}\lambda}$)

$$\mathbf{A}_{\mathbf{k}} = \sum_{\lambda} \mathbf{u}_{\mathbf{k}\lambda} A_{\mathbf{k}\lambda}. \tag{1.68}$$

For simplicity, we assume that the polarization vectors are real and $\mathbf{u}_{\mathbf{k}\lambda} = \mathbf{u}_{-\mathbf{k}\lambda}$. The complex number $A_{\mathbf{k}\lambda}$ still encodes two degrees of freedom that are related to the plane waves with wave vectors $\mathbf{k}$ and $-\mathbf{k}$. This is apparent from the relation

$$\mathbf{A}^*_{\mathbf{k}} = \mathbf{A}_{-\mathbf{k}} \tag{1.69}$$

that follows from the fact that $\mathbf{A}(\mathbf{x}, t)$ is real. We now make the decomposition

$$A_{\mathbf{k}\lambda} = q_1 + \mathrm{i} q_2, \quad A_{-\mathbf{k}\lambda} = q_1 - \mathrm{i} q_2, \tag{1.70}$$

where the indices $\mathbf{k}\lambda$ have been dropped just for the clarity of presentation. In terms of these variables, the terms with $\mathbf{k}$ and $-\mathbf{k}$ combine to give a sum of two harmonic oscillators:

$$L_{\mathbf{k}} = \varepsilon_0 \left(\dot{q}_1^2 + \dot{q}_2^2 - c^2 k^2 \left(q_1^2 + q_2^2 \right) + \text{interactions} \right) \tag{1.71}$$

(The factor $\frac{1}{2}$ drops out because both $\mathbf{k}$ and $-\mathbf{k}$ give the same contribution.)

Quantization. Canonical quantization means that the coordinate q_1 and its canonically conjugate momentum

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = 2\varepsilon_0 \dot{q}_1 \tag{1.72}$$

are replaced by operators with the commutator $[p_\alpha, q_\beta] = -\mathrm{i}\hbar\, \delta_{\alpha\beta}$ ($\alpha, \beta = 1, 2$). The Hamiltonian is given by

$$H = \frac{1}{4\varepsilon_0} \left(p_1^2 + p_2^2 \right) + \varepsilon_0 c^2 k^2 \left(q_1^2 + q_2^2 \right) + \text{interaction} \tag{1.73}$$

so that we are dealing with two uncoupled harmonic oscillators. We can use the well-known results from the Quantum Mechanics lecture if we write $\omega = ck$ and $m = 2\varepsilon_0$. Here ω is really a frequency (the light frequency for a plane wave), and m is not a mass because q is not a position coordinate, but has the dimensions of a vector potential.

The quantization of the harmonic oscillator is most easily written down in terms of the annhilation and creation operators a_α and $a_\alpha^\dagger$. These are dimensionless quantities with commutator $\left[a_\alpha, a_\beta^\dagger\right] = \delta_{\alpha\beta}$. They allow to represent coordinates and momenta as

$$q_\alpha = \sqrt{\frac{\hbar}{4\varepsilon_0\omega}} \left(a_\alpha + a_\alpha^\dagger \right), \tag{1.74}$$

$$p_\alpha = -\mathrm{i}\sqrt{\hbar\varepsilon_0\omega} \left(a_\alpha - a_\alpha^\dagger \right). \tag{1.75}$$

We thus get the following expansion of the vector potential

$$\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{k},\lambda}{}' \sqrt{\frac{\hbar}{4\varepsilon_0 V \omega_{\mathbf{k}}}} \mathbf{u}_{\mathbf{k}\lambda} \left\{ \left(a_1 + a_1^\dagger + \mathrm{i}\left(a_2 + a_2^\dagger \right) \right) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \left(a_1 + a_1^\dagger - \mathrm{i}\left(a_2 + a_2^\dagger \right) \right) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \right\} \tag{1.76}$$

The primed sum is to remind us that we sum only over 'one half' of k-space, regrouping $\mathbf{k}$ and $-\mathbf{k}$. We now introduce the operators

$$b_1 = \frac{a_1 + \mathrm{i}a_2}{\sqrt{2}}, \quad b_2 = \frac{a_1 - \mathrm{i}a_2}{\sqrt{2}}. \tag{1.77}$$

They also satisfy the commutation relations $\left[b_\alpha,\, b_\beta^\dagger\right] = \delta_{\alpha\beta}$ and are therefore equally well suited to represent the normal modes of the field. They have the advantage that the plane wave expansion (1.76) can be written in the form

$$\mathbf{A}(\mathbf{x}) \;=\; \sum_{\mathbf{k},\lambda}{}' \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\mathbf{k}}}\mathbf{u}_{\mathbf{k}\lambda}\left\{b_1 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + b_2 \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.}\right\}. \tag{1.78}$$

We can now label the mode operators as $b_1 = a_{\mathbf{k}\lambda}$ and $b_2 = a_{-\mathbf{k}\lambda}$, so that the summation can be extended over all k-space:

$$\boxed{\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\mathbf{k}}}\left\{\mathbf{u}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.}\right\}.} \tag{1.79}$$

This is the standard expansion of the vector potential in quantum electrodynamics.

Interaction and free field Hamiltonian. The interaction Hamiltonian with the transverse current takes the following form in the quantized theory:

$$H_{FA} \;=\; \int \mathrm{d}^3x\, \mathbf{A}(\mathbf{x}) \cdot \mathbf{j}_\perp(\mathbf{x}, t) \tag{1.80}$$

$$\;=\; \sum_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_\mathbf{k}}}\left\{a_{\mathbf{k}\lambda} \mathbf{u}_{\mathbf{k}\lambda} \cdot \mathbf{j}^*_{\perp,\mathbf{k}}(t) + \mathrm{h.c.}\right\}, \tag{1.81}$$

where $\mathbf{j}_{\perp,\mathbf{k}}(t)$ is the spatial Fourier transform.

The dynamics of the field itself is generated by the well-known collection of harmonic oscillators:

$$H_F = \sum_{\mathbf{k},\lambda} \hbar\omega_\mathbf{k} \left(a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \tfrac{1}{2}\right). \tag{1.82}$$

In the present context, we treated the current density as an 'external', given function with an explicit time-dependence. For this reason, there is no Hamiltonian required for it. If the motion of the charges is quantized as well, one needs, of course, an 'atomic' Hamiltonian to generate their dynamics.

Field commutators. From the mode expansions (1.79, 1.64, 1.65) and the commutation relations between the annihilation operators, one can compute the commutator between the electric and magnetic fields. In the Lagrangian description, one finds already that the vector potential and the electric field are (up to a factor $-\varepsilon_0$) canonically conjugate variables. This is only true in the space of 'transverse vector functions' (fields with zero divergence), however. In the quantized theory, the corresponding commutator is

$$[A_l(\mathbf{x}, t),\, E_m(\mathbf{x}', t)] = \frac{\mathrm{i}\hbar}{\varepsilon_0}\delta^\perp_{lm}(\mathbf{x} - \mathbf{x}') \tag{1.83}$$

where $\delta^\perp_{lm}$ is the 'transverse δ-function' defined in Eq.(1.45). As discussed above, this distribution acts like a δ-function on fields with zero divergence, and projects an arbitrary vector field $\mathbf{F}$ on its transverse part $\mathbf{F}^\perp$

$$F^\perp_l(\mathbf{x}) = \int \mathrm{d}^3x'\, \delta^\perp_{lm}(\mathbf{x} - \mathbf{x}')\, F_m(\mathbf{x}') \tag{1.84}$$

By definition, the transverse part has zero divergence, $\nabla \cdot \mathbf{F}^\perp = 0$. The projection is most easily constructed in Fourier space

$$F^\perp_j(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{k^2\delta_{jl} - k_j k_l}{k^2} \tilde{F}_l(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{1.85}$$

where $\tilde{\mathbf{F}}(\mathbf{k})$ is the spatial Fourier transform of $\mathbf{F}(\mathbf{x})$. This relation allows to deduce explicit expressions for the transverse δ-function.

Note that in the wave equation (1.7) for the vector potential, we have an alternative relation between the current density $\mathbf{j}$ and its transverse part:

$$\mathbf{j}_\perp = \mathbf{j} - \varepsilon_0 \partial_t \nabla\phi,$$

The 'longitudinal part' that is subtracted here is thus related to the electrostatic field created by the corresponding charge density. More details are discussed in the exercises.

1.2 Photons and the quantum vacuum

1.2.1 'Photons'

The present quantized description of the electromagnetic field allows us to give a more precise meaning to the word 'photon'. *A photon is an ex-*

citation of a mode of the field. We have seen that the quantized field reduces to a collection of harmonic oscillators, one for each mode. As we know from the harmonic oscillator, its stationary states are labelled by non-negative numbers $n = 0, 1, \ldots$. One says that in these states, the mode contains 'n photons'. The creation operator $a^\dagger$ whose action on these states is $a^\dagger|n\rangle \propto |n+1\rangle$, thus 'creates one photon'. This picture is consistent with the assumption (dating back to Einstein (1905)) that photons correspond to 'energy packets' of $\hbar\omega_k$ of the electromagnetic field. If a plane wave mode expansion is used, we can also say that the momentum of a photon is $\hbar\mathbf{k}$, as we know from de Broglie (1926) or from the Compton effect.

It is however possible to use different mode expansions for the same field. For example, we could have used an expansion in terms of spherical vector harmonics which differs from the plane wave expansion by a (infinite-dimensional) unitary transformation. A single-photon state in the plane wave basis thus becomes a superposition of single-photon states in infinitely many spherical modes. Conversely, a 'photon' in this description would not correspond to a plane wave (its momentum would not be $\hbar\mathbf{k}$), but it would have a definite angular momentum with respect to the origin. It is even possible to define photons that are wavepackets localized in time, by superposing plane waves with neighboring frequencies. This picture allows to describe experiments with 'single-photon pulses'. We refer to the exercises to look at these properties in more detail.

Finally, we precall that the computation of a suitable set of mode functions is a 'classical' problem: no quantum theory is needed to state it. Using a cubic 'quantization' box, one can show that the plane waves with wave vectors $\mathbf{k} = (2\pi/L)(n_x, n_y, n_z)^T$ ($n_i \in \mathbb{Z}$) are orthogonal with respect to the scalar product

$$\int_{L^3} \mathrm{d}^3x\, \mathbf{F}^*(\mathbf{x}) \cdot \mathbf{G}(\mathbf{x}). \tag{1.86}$$

This is clear for different wave vectors, $\mathbf{k} \neq \mathbf{k}'$. But for a given $\mathbf{k}$, one can also find two orthogonal polarization vectors $\boldsymbol{\varepsilon}_{1,2}$ that give orthogonal modes (the scalar product $\boldsymbol{\varepsilon}_1^* \cdot \boldsymbol{\varepsilon}_2$ is zero). The corresponding magnetic field mode functions are also orthogonal. The electrodynamics becomes a 'quantum' theory only when the amplitudes of the mode functions become suitably normalized operators.

1.2.2 Vacuum fluctuations

In the vacuum state, the expectation value $\langle \mathbf{E}(\mathbf{x}, t)\rangle = 0$ because a_κ annihilates the vacuum state and $a_\kappa^\dagger$ can be made to act to the left on the vacuum state which is annihilated.

Note. The same is true for any stationary (or number) state, see Sec.1.3.1 below.

The vacuum fluctuations become visible in the next moment of the field: for a given mode κ,

$$\begin{aligned}\frac{\hbar\omega_\kappa}{2\varepsilon_0}\langle(\mathbf{f}_\kappa(\mathbf{x})a(t) + \text{h.c.})^2\rangle &= \frac{\hbar\omega_\kappa}{2\varepsilon_0}|\mathbf{f}_\kappa(\mathbf{x})|^2\langle a_\kappa(t)a_\kappa^\dagger(t)\rangle \\ &= \frac{\hbar\omega_\kappa}{2\varepsilon_0}|\mathbf{f}_\kappa(\mathbf{x})|^2\end{aligned} \tag{1.87}$$

in the last step, we have used that $a_\kappa(t) = a_\kappa(0)\,\mathrm{e}^{-\mathrm{i}\omega_\kappa t}$. In the case of plane wave modes, we have $|\mathbf{f}_\kappa(\mathbf{x})|^2 = 1/V$. The sum over wave vectors $\mathbf{k}$ and polarization indices λ can be written in the form

$$\langle \mathbf{E}^2(\mathbf{x}, t)\rangle = \int\limits_0^\infty \frac{\mathrm{d}\omega}{2\pi}\frac{\hbar\omega_\kappa}{2\varepsilon_0}\rho(\omega) \tag{1.88}$$

where $\rho(\omega)$ is the so-called 'density of modes' (per unit frequency and unit volume):

$$\rho(\omega) = \frac{2\pi}{V}\sum_{\mathbf{k}\lambda}\delta(\omega - \omega_{\mathbf{k}\lambda}) \tag{1.89}$$

For the plane wave modes, $\omega_{\mathbf{k}\lambda}$ only depends on the magnitude of $\mathbf{k}$, and in the continuum limit,

$$\sum_{\mathbf{k}} = V\int\frac{\mathrm{d}^3k}{(2\pi)^3} \tag{1.90}$$

we get after a simple integration

$$\rho(\omega) = \frac{2}{\pi}\frac{\omega^2}{c^3} \tag{1.91}$$

The full vacuum fluctuation of the electric field is then infinite

$$\langle \mathbf{E}^2(\mathbf{x}, t)\rangle = \frac{\hbar}{\pi\varepsilon_0 c^3}\int\limits_0^\infty\frac{\mathrm{d}\omega}{2\pi}\omega^3 \tag{1.92}$$

because the integral diverges at the upper limit. This correlates with an infinite electromagnetic energy density in vacuum (multiply with $\varepsilon_0/2$ and add the magnetic component, which doubles the result). This "infinite vacuum energy" is one of the unresolved problems in physics. "Intuitive cutoffs" at short wavelengths, for example, at the Planck scale ($10^{-32}\,\mathrm{m}$) give a finite energy density, but with a value that differs by something like 100 orders of magnitude from the energy density associated with cosmological observations (including "dark energy", "cosmological constants" and so on).

Exercise. Find a cutoff such that the vacuum energy density equals the 'critical density of the Universe' (the critical mass density is roughly $10^{-29}\,\mathrm{g/cm}^3$, given the current expansion rate of the Universe.)

A finite value can be found, if we compute an autocorrelation function of the electric field. A similar calculation gives

$$\langle \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t')\rangle = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\frac{\hbar\omega_\kappa}{2\varepsilon_0 c^3}\rho(\omega)\,\mathrm{e}^{-\mathrm{i}\omega(t-t')} \tag{1.93}$$

This leads to the integral representation of the Γ-function in the complex plane and finally to

$$\langle \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t')\rangle = \frac{3\hbar}{\pi^2\varepsilon_0 c^3\,\tau^4} \tag{1.94}$$

which is finite for all $\tau = t - t' \neq 0$. For any finite value of τ, the electric vacuum energy is thus of the order of $\hbar\omega_\tau/\lambda_\tau$ with the chacteristic frequency $\omega_\tau = 1/\tau$ and wavelength $\lambda_\tau = c\tau$.

1.2.3 Casimir energy

The Casimir force is the attraction between two metallic mirrors placed in vacuum. It is interpreted in terms of the change in the zero-point energy (the famous $\frac{1}{2}\hbar\omega$ of the harmonic oscillator ground state) induced by the presence of the mirrors. We give here an sketch of the calculation done by Casimir around 1948 [Proc. Kon. Ned. Akad. Wet. 51 (1948) 793].

We consider the ground state energy of the multi-mode electromagnetic field

$$E_0 = \sum_{\mathbf{k}\lambda}\frac{\hbar\omega_{\mathbf{k}\lambda}}{2}$$

that is of course infinite and compare the cases of a planar cavity formed by two mirrors (distance L) and empty space (i.e., two mirrors infinitely apart). In the first case, we have standing wave modes between the mirrors with a frequency

$$\omega^{(cav)} = c\sqrt{K^2 + k_n^2}, \qquad k_n = \frac{n\pi}{L}$$

with $K^2 = k_x^2 + k_y^2$ and $n = 1, 2, \ldots$, while in empty space,

$$\omega = c\sqrt{K^2 + k_z^2}$$

with $-\infty < k_z < \infty$. We first compute the difference in the electromagnetic mode density per volume AL where A is the 'quantization area' in the xy-plane. We cheat with the polarizations and multiply by a factor 2:

$$\begin{aligned}\rho_L(\omega) &= \frac{4\pi}{AL}\sum_{\mathbf{K},n}\delta\Big(\omega - c\sqrt{K^2+k_n^2}\Big)\\ &= \frac{2}{L}\int_0^\infty K\,\mathrm{d}K\,\sum_n \delta\Big(\omega - c\sqrt{K^2+k_n^2}\Big)\end{aligned} \tag{1.95}$$

The integration over K can be performed with the substitution $K \mapsto c\sqrt{K^2+k_n^2}$ and gives

$$\rho_L(\omega) = \frac{2\omega}{Lc^2}\sum_n \Theta(\omega - ck_n) \tag{1.96}$$

where Θ is the step function. It arises because for a given n, there are no modes with frequency smaller than ck_n. The same calculation in the infinite volume gives

$$\rho_\infty(\omega) = \frac{2\omega}{c^2}\int_0^\infty \frac{\mathrm{d}k_z}{\pi}\Theta(\omega - ck_z) \tag{1.97}$$

The k_z integral can of course be performed, but we keep it here to illustrate one of the basic features of the Casimir calculation: the result originates from the difference between a 'discrete spectrum' (the sum over the k_n) and a continuum (the integral over k_z).

The Casimir energy is now found as the difference in vacuum energy *per area* in the space of length L between the mirrors:

$$\Delta E = L\int_0^\Omega \frac{\hbar\omega}{2}\left(\rho_L(\omega) - \rho_\infty(\omega)\right) \tag{1.98}$$

We have introduced an upper cutoff frequency Ω because the integrals are likely to diverge in the UV. One of the mathematical difficulties (that we are not going to discuss here) is to what extent the results depend on the cutoff. At a suitable stage of the calculation, we are going to take the limit $\Omega \to \infty$, of course.

The ω-integrals can be performed before the sum over n (the integral over k_z), and one gets

$$\Delta E = \frac{\hbar}{6\pi c^2} \left[\sum_{n=1}^{\lfloor \Omega L/\pi c \rfloor} \left(\Omega^3 - (ck_n)^3 \right) - \int_0^{\Omega/c} \frac{\mathrm{d}k_z}{\pi} \left(\Omega^3 - (ck_z)^3 \right) \right] \tag{1.99}$$

where $\lfloor x \rfloor$ is the largest integer smaller than x. Introducing the number $N = \Omega L/\pi c$ and the dimensionless integration variable $z = k_z L/\pi$, this can be written in the form

$$\Delta E = \frac{\hbar c \pi^2}{6L^3} \left[\sum_{n=1}^{\lfloor N \rfloor} \left(N^3 - n^3 \right) - \int_0^N \mathrm{d}z \left(N^3 - z^3 \right) \right] \tag{1.100}$$

The difference in brackets is some magic number and equal to $-1/120$ in the limit $N \to \infty$. (A proof is sketched below). The Casimir energy of two mirrors is thus equal to

$$\Delta E = -\frac{\hbar c \pi^2}{720 L^3} \tag{1.101}$$

so that the force per unit area is $F_C/A = -\hbar c \pi^2/240 L^4$: since the energy decreases as $L \to 0$, the two mirrors placed in vacuum attract each other.

Note that this result is independent of the nature of the mirrors, as well as their electric charge. The electromagnetic field only enters inasmuch as its modes give a contribution to the energy of the vacuum state. Field theorists have computed the contribution to the Casimir energy from the Dirac electron field, for example. It is small if the mirror separation is large compared to the Compton wavelength $\hbar/mc \approx 2.5\,\mathrm{pm}$ — which is nearly always the case. The Casimir energy, being attractive, is sometimes thought of a means to 'stabilize' a classical model of the electron (a bag of charge) against the Coulomb repulsion.

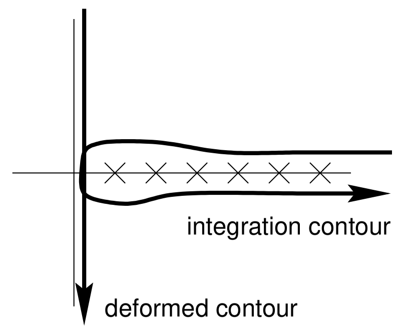

Figure 1.1: Integration contour for (1.103).

Sum minus integral

We use a trick in the complex plane. There is a theorem for functions f and D that are analytical in a domain limited by the integration contour $\mathcal{C}$:

$$\frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{C}} \mathrm{d}z\, f(z) \frac{\mathrm{d}}{\mathrm{d}z} \log D(z) = \sum_n f(z_n) \tag{1.102}$$

where the z_n are the zeros of D in the interior of the contour. We will use $f(z) = N^3 - z^3$ and choose $D(z)$ such that it is zero for the values $z_n = n$: $D(z) = \sin(\pi z)$. The differentiation under the integral sign gives

$$\sum_{n=0}^{N} (N^3 - n^3) = \frac{1}{2} \oint_{\mathcal{C}_N} \mathrm{d}z\, (N^3 - z^3) \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} \tag{1.103}$$

We chose an integration contour as shown in fig. 1.1 running from $+N$ above the real axis to 0 and going back to $+N$ below the real axis (the sum over all positive zeros of $\sin \pi z$ thus gives the sum on the left hand side). We are eventually interested in the limit $N \to \infty$. Make the following transformations on the upper and lower part of the contour:

$$\text{upper part:} \quad \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} = -1 + \frac{2\mathrm{e}^{\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

$$\text{lower part:} \quad \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} = 1 + \frac{2\mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

The constants ± 1 give for both the upper and lower path an integral over $N^3 - z^3$ that can be combined into

$$\sum_{n=0}^{N}(N^3 - n^3) \;=\; \int_0^N \mathrm{d}z\,(N^3 - z^3) + \oint_C \mathrm{d}z\,(N^3 - z^3)\frac{\mathrm{e}^{\pm \mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

In the second integral, the exponential takes the appropriate sign on the upper and lower parts of the contour. The first integral on the right hand side is exactly the integral that we have to subtract in Eq.(1.100). The upper and lower parts of the contour can now be shifted onto the (positive or negative) imaginary axis because the integrand has no singularities (these are located on the real axis only). The quarter-circle with radius $|z| = N$ contributes only a negligible amount because of the $\mathrm{e}^{\pm \mathrm{i}\pi z}$.

Choosing $z = \pm \mathrm{i}t$ on the imaginary axis, we get

$$\begin{aligned}
&\oint_C \mathrm{d}z\,(N^3 - z^3)\frac{\mathrm{e}^{\pm \mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} \\
&= -\mathrm{i}\int_0^\infty \frac{\mathrm{d}t\,[N^3 - (\mathrm{i}t)^3]\,\mathrm{e}^{-\pi t}}{\mathrm{e}^{-\pi t} - \mathrm{e}^{\pi t}} - \mathrm{i}\int_0^\infty \frac{\mathrm{d}t\,[N^3 - (-\mathrm{i}t)^3]\,\mathrm{e}^{-\pi t}}{\mathrm{e}^{\pi t} - \mathrm{e}^{-\pi t}} \\
&= -2\int_0^\infty \frac{\mathrm{d}t\,t^3}{\mathrm{e}^{2\pi t} - 1}
\end{aligned}$$

Note that the imaginary parts of the two integrals that involve N^3 cancel each other: we have finally eliminated the cutoff.

You have encountered the last integral in the context of blackbody radiation. Changing to the integration variable $t' = 2\pi t$, the integral gives $1/240$, so that we have in the end

$$\lim_{N\to\infty}\left(\sum_{n=0}^{N}(N^3 - n^3) - \int_0^N \mathrm{d}z\,(N^3 - z^3)\right) = -\frac{2}{240} = -\frac{1}{120} \qquad (1.104)$$

as announced in the text.

1.3 States of the quantized single-mode field

The quantized light field can be in different states. We start here with a single mode of the field. This may be a oversimplification, but single-mode fields have become part of the experimental reality with the advent

of high-quality optical cavities. These devices give an electromagnetic field whose amplitude, in the region between two well-reflecting mirrors, is much higher at some resonant frequencies. The 'mode function' is in this case not a plane wave, of course, but a standing wave. In the transverse directions, one often has a gaussian profile. Around a cavity resonance, it is a frequent approximation to treat the full field as if it contained only a single mode. The coupling to other modes may be taken into account as a loss.

The electric field is given by

$$\mathbf{E}(\mathbf{x}, t) = E_1 \boldsymbol{\varepsilon}\left(a(t) + a^\dagger(t)\right)\sin kz \qquad (1.105)$$

where z is the coordinate along the cavity axis and $k = n_z\pi/L$. The factor E_1 can be called the 'electric field per photon'. From the multimode expansion, E_1 is given by the prefactor $E_k = (\hbar\omega_k/2\varepsilon_0 V)^{1/2}$. The corresponding 'intensity' is

$$I_{1\mathrm{ph}} = \varepsilon_0 c E_k^2 = \frac{\hbar\omega_k c}{2V}. \qquad (1.106)$$

In a cavity, we can take for V the volume 'filled' by the mode. For a transverse mode size of 1 micrometer and a cavity length of 1 cm, we get $I_{1\mathrm{ph}} \sim 10^3\,\mathrm{mW/cm}^2$ which is not really small. The total power, however, is quite small: about $10^{-8}\,\mathrm{W}$. Note also that these numbers are based on very 'tight' (diffraction-limited) focussing — beams with larger cross-section have a smaller 'field per photon'.

In the Heisenberg picture, the field operator evolves as

$$\mathbf{E}(\mathbf{x}, t) = E_1 \boldsymbol{\varepsilon}\left(a\,\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{-\mathrm{i}\omega t}\right)\sin kz \qquad (1.107)$$

A combination of annihilation and creation operators like the one in parentheses is called a 'quadrature'. Quadratures always come in pairs. One can find a second quadrature variable by shifting the origin of time by one quarter period: $\propto -\mathrm{i}\,a\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{i}\,a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}$. This corresponds to the magnetic field (compare eqs. 1.64 and 1.65). In analogy to the harmonic oscillator, one often uses the following quadrature variables

$$X = \frac{a + a^\dagger}{\sqrt{2}} \qquad P = \frac{a - a^\dagger}{\sqrt{2}\,\mathrm{i}} \qquad (1.108)$$

The simplest quantum states of the single mode field are given by the well-known stationary states of the harmonic oscillator.

1.3.1 Number states

These quantum states are called 'Fock states' or 'number states' $|n\rangle$. They are eigenstates of the 'photon number operator'

$$\hat{n} = a^\dagger a = a a^\dagger - 1 \tag{1.109}$$

and are generated by applying the creation operator to the ground state of the oscillator:

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \tag{1.110}$$

The expectation value of the annihilation operator is zero in a number state:

$$\langle a \rangle_n = \langle n | a | n \rangle = \sqrt{n} \langle n | n - 1 \rangle = 0 \tag{1.111}$$

The same is true for the creation operator. It follows that the electric field average vanishes in a Fock state:

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle_n = 0 \tag{1.112}$$

The ground state of the field mode is called the 'vacuum' (no photon, i.e., no excitation present). It is found by looking for the state that is annihilated by the annihilation operator: $a|\mathrm{vac}\rangle = 0$. Obviously, this is also an eigenstate of the photon number operator with zero photons: $|\mathrm{vac}\rangle = |0\rangle$. In the vacuum state, the electric field is also zero on average, of course.

But there are fluctuations around this average, called 'quantum noise'. In the vacuum state of the single mode (1.105), e.g., we get

$$\langle \mathbf{E}(\mathbf{x}, t)^2 \rangle_0 = E_1^2 \sin^2 kz \, \langle 0 | \left(a(t) + a^\dagger(t) \right) \left(a(t) + a^\dagger(t) \right) | 0 \rangle \tag{1.113}$$

and this combination of operators gives an average

$$\langle 0 | \left(a(t) + a^\dagger(t) \right) \left(a(t) + a^\dagger(t) \right) | 0 \rangle = \langle 0 | a(t) a^\dagger(t) | 0 \rangle = 1 \tag{1.114}$$

The 'vacuum noise' in our mode is thus given by the squared single photon field $E_1^2 \sin^2 kz$. Similarly, the other quadrature variable $a(t) - a^\dagger(t)$ shows a noise strength of unity. This is in accordance with Heisenberg's indeterminacy relation, since

$$\left[a(t) + a^\dagger(t), \, a(t) - a^\dagger(t) \right] = -2. \tag{1.115}$$

In the exercises, you are asked to compute the variances of these quadratures in an arbitrary number state $|n\rangle$.

The electric field fluctuations in the vacuum state play an important role in the interpretation of the Casimir effect and of the spontaneous decay of excited atomic states.

Finally, the creation and annihilation operators connect states whose photon numbers differ by one. In this sense, the 'creation operator' $a^\dagger$ creates one photon since for example

$$\langle 1 | a^\dagger | 0 \rangle = 1. \tag{1.116}$$

This matrix element plays an important role when one computed the probability amplitude that an excited atomic state emits a photon. For stimulated emission, one needs $\langle n+1 | a^\dagger | n \rangle = \sqrt{n+1}$. Similarly, the 'annihilator' a destroys one photon:

$$\langle 0 | a | 1 \rangle = 1. \tag{1.117}$$

This matrix element is needed to compute absorption, and in the general case, $\langle n - 1 | a | n \rangle = \sqrt{n}$.

1.3.2 Coherent states

The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{1.118}$$

Since a is not an hermitean operator, α can be complex. *In a coherent state, the average electric field is nonzero:*

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle_\alpha = E_1 \sin kz \, \langle \alpha | \left(a(t) + a^\dagger(t) \right) | \alpha \rangle = E_1 \sin kz \, \left(\alpha \, \mathrm{e}^{-\mathrm{i}\omega t} + \alpha^* \, \mathrm{e}^{\mathrm{i}\omega t} \right). \tag{1.119}$$

We have assumed the field in a coherent state of the initial annihilator a. This expression is the same that we have used in chapter 1 for a classical, monochromatic field. The magnetic field quadrature also has on average its classical value in a coherent state. Coherent states are thus very useful to represent laser fields. We see that α measures the electric field strength in units of the 'single photon field' E_1. If we compute the average photon number in a coherent state, we get

$$\langle \hat{n} \rangle_\alpha = \langle a | a^\dagger a | \alpha \rangle = |\alpha|^2, \tag{1.120}$$

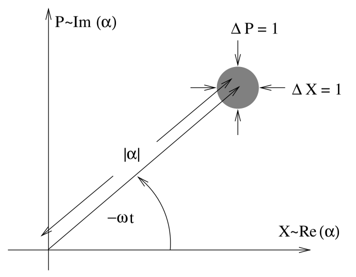

Figure 1.2: Representation of a coherent state in phase space.

so that as an order of magnitude $\langle E\rangle \approx E_1\langle\hat{n}\rangle^{1/2}$ (note the nonlinear dependence).

Coherent states are not stationary, but rotate in the complex α-plane: if $|\psi(0)\rangle = |\alpha\rangle$, then $|\psi(t)\rangle = |\alpha\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle$. This can be shown using the expansion of a coherent state in terms of number states:

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{1.121}$$

Note that number states with arbitrarily high photon numbers are present in a coherent state. More specifically, we can introduce the probability $p_n(\alpha)$ of finding n photons in a coherent state:

$$p_n(\alpha) = |\langle n|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{1.122}$$

which is a 'Poisson distribution' (the probability distribution of the sum of independent random bits). In the exercises, you are asked to compute the average photon number and its fluctuations (variance) in a coherent state.

The field quadratures also show quantum fluctuations around their classical average in a coherent state. This is inevitable because of the Heisenberg inequality. In the exercises, you are asked to show that these are equal to the quantum noise in the vacuum state (which is in fact a particular coherent state with $\alpha = 0$). This result is often graphically displayed in the complex α-plane by the sketch shown in fig. 1.2. Note that since $a = (X + \mathrm{i}P)/\sqrt{2}$, the α-plane may be identified with the classical phase

space of a harmonic oscillator. The gray area in this sketch indicates values for the position and momentum quadratures that are probable outcomes of measurements. This representation is of course schematic since X and P cannot be measured simultaneously. We shall give it a precise meaning in chapter 5 where we show how coherent states can be used to expand any field state. (There are some subtleties related to the fact that they are not eigenstates of an hermitean operator.)

How would a number state look in this sketch? A first guess is a 'ring', since the photon number (or energy) is fixed and shows no fluctuations. This is not far from the precise answer that we shall see in chapter 5.

Generation of coherent states. Finally, how is it possible to generate a coherent state physically? To this end, consider the single-mode Hamiltonian

$$H = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) + \mathrm{i}\hbar g\left(\mathrm{e}^{-\mathrm{i}\omega t}\,a^\dagger - \mathrm{e}^{\mathrm{i}\omega t}\,a\right) \tag{1.123}$$

The interaction describes, e.g., the coupling of a classical dipole oscillator with the field mode (the dipole operators $\sigma_\pm$ are replaced by c-numbers).

In the interaction representation, the first term is transformed away and the exponentials $\mathrm{e}^{\pm\mathrm{i}\omega t}$ disappear. Starting with the mode in the vacuum state, we thus find using the Baker-Hausdorff-Campbell identity

$$|\psi(t)\rangle = \exp\left\{gt\left(a^\dagger - a\right)\right\}|0\rangle = \mathrm{e}^{-g^2t^2/2}\,\mathrm{e}^{gt\,a^\dagger}\,\mathrm{e}^{-gt\,a}|0\rangle \tag{1.124}$$

Now the annihilation operator gives 0 when acting on the vacuum state, so that its exponential reduces to unity here. Expanding the exponential with the creation operator in a power series, we find

$$|\psi(t)\rangle = \mathrm{e}^{-g^2t^2/2}\sum_{n=0}^{\infty}\frac{(gt)^n}{\sqrt{n!}}|n\rangle = |gt\rangle \tag{1.125}$$

This interaction thus generates a coherent state with amplitude $\alpha = gt$ that grows linearly in time. To obtain a stationary result, either the 'oscillator amplitude' g can be made time-dependent, or loss processes have to be added.

We have just shown that coherent states can be obtained by applying a 'displacement operator' to the vacuum state:

$$|\alpha\rangle = D(\alpha)|0\rangle \qquad D(\alpha) = \exp\left\{\alpha a^\dagger - \alpha^* a\right\} \tag{1.126}$$

This unitary operator also displaces the creation and annihilation operators as follows (prove it yourself)

$$D^\dagger(\alpha)\, a\, D(\alpha) \;=\; a + \alpha \qquad (1.127)$$

$$D^\dagger(\alpha)\, a^\dagger\, D(\alpha) \;=\; a^\dagger + \alpha^*. \qquad (1.128)$$

This identity is useful to show that the field quadrature fluctuations in a coherent state are those of the vacuum state.

1.3.3 Squeezed states

You should have got the feeling up to now that the quantized field essentially differs from a classical field by its ('quantum') fluctuations. So people have thought whether it is possible to reduce the quantum noise in a field quadrature to get something even 'more classical' – or having less noise. This can be achieved in part, to 50%, say. Of course, one cannot beat the Heisenberg inequality, and the reduced fluctuations in one quadrature have to be paid by enhanced noise in the other one.

In the exercises, you are asked to show that the following operator

$$S(\xi) = \exp\left(\xi a^{\dagger 2} - \xi^* a^2\right) \qquad (1.129)$$

'squeezes' the fluctuations of the vacuum state such that one quadrature component (fixed by the phase of the complex number ξ) has less quantum noise. Note that a 'general quadrature' variable may be defined by

$$X_\theta = \frac{a\, \mathrm{e}^{\mathrm{i}\theta} + a^\dagger\, \mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{2}} \qquad (1.130)$$

Two 'orthogonal' quadratures are then given by X_θ and $X_{\theta+\pi/2}$. The usual position and momentum quadratures correspond to $\theta = 0, \pi/2$. In the state $S(\xi)|0\rangle$ with $\xi = |\xi|\, \mathrm{e}^{-2\mathrm{i}\theta}$, the uncertainties of the quadratures are

$$\Delta X_\theta^2 = \frac{\mathrm{e}^{2|\xi|}}{2}, \qquad \Delta X_{\theta+\pi/2}^2 = \frac{\mathrm{e}^{-2|\xi|}}{2}, \qquad (1.131)$$

their product being unchanged.

Note also that the 'squeezed vacuum' $S(\xi)|0\rangle$, when expanded in the Fock basis, contains only even photon number states:

$$S(\xi)|0\rangle = \sum_{m=0}^{\infty} c_{2m} |2m\rangle. \qquad (1.132)$$

This is because the squeezing operator (1.129) only contains evens powers of the creation operator.

Exercise: compute the coefficients c_{2m}. Answer (details may be wrong):

$$c_{2m} = N \left(\frac{(2m-1)!!}{(2m)!!} \right)^{1/2} \mathrm{e}^{-2\mathrm{i}m\phi}\, \tanh^m r$$

where N is a normalization, $\xi = (r/2)\mathrm{e}^{2\mathrm{i}\phi}$, and $n!!$ is the product $n(n-2)\cdots$ of all numbers with the same parity up to n.

Squeezing from a nonlinear medium. The 'squeezing' operator (1.129) can be realized with the interaction Hamiltonian

$$H_{\mathrm{int}} = \mathrm{i}\hbar \left(g\, \mathrm{e}^{-2\mathrm{i}\omega t}\, a^{\dagger 2} - g^*\, \mathrm{e}^{2\mathrm{i}\omega t}\, a^2 \right) \qquad (1.133)$$

with the squeezing parameter given by $\xi = \int \mathrm{d}t\, g(t)$. This interaction occurs in nonlinear optics. To get a qualitative understanding, imagine a medium with a field-dependent dielectric constant ('$\chi^{(2)}$ nonlinearity'). This is usually forbidden for symmetry reasons, but it happens in some special cases. In the electromagnetic energy density, one has

$$u = \frac{\varepsilon(|\mathbf{E}|)}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \qquad (1.134)$$

where the linearization

$$\varepsilon(|\mathbf{E}|) = \varepsilon_0 \left(1 + n_2|\mathbf{E}|\right)^2 \approx \varepsilon_0 \left(1 + 2n_2|\mathbf{E}|\right)$$

is often appropriate. In the quantum picture, this gives a contribution to the Hamiltonian with a term of third order in the field:

$$H_3 = \varepsilon_0 n_2 \int_V \mathrm{d}^3x\, |\mathbf{E}(\mathbf{x}, t)|^3 \qquad (1.135)$$

Let us now pick out two spatial modes of the field and put one of it into a coherent state $|\alpha\, \mathrm{e}^{-\mathrm{i}\omega_p t}\rangle$ with a 'large' amplitude $|\alpha| \gg 1$. The index 'p' is for 'pump field'. Let us call the other mode (the 'quantum' one) the 'signal'. The electric field is then

$$\mathbf{E}(\mathbf{x}, t) = E_p a_p \boldsymbol{\varepsilon}_p\, \mathrm{e}^{-\mathrm{i}(\omega_p t - \mathbf{k}_p \cdot \mathbf{x})} + E_1 \boldsymbol{\varepsilon} a(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \text{h.c.} \qquad (1.136)$$

The interaction Hamiltonian thus generates cross terms of the form[1]

$$H_{\text{int}} = \ldots + \hbar\left(g\,e^{-i\omega_p t}\,a_p a^{\dagger 2} + g^*\,e^{i\omega_p t}\,a_p^\dagger a^2\right) \tag{1.137}$$

$$\hbar g = 3\varepsilon_0 n_2 E_p E_1 \boldsymbol{\varepsilon}_p \cdot \boldsymbol{\varepsilon}^* \int_V d^3x\, e^{i(\mathbf{k}_p - 2\mathbf{k})\cdot\mathbf{x}} \tag{1.138}$$

One often ignores the quantum fluctuations of the pump mode and replaces its annihilation operator a_p by the coherent state amplitude α. The interaction (1.137) then looks quite like our model Hamiltonian (1.133).

The nonlinear squeezing parameter $g\alpha$ is nonzero when the pump and signal modes are 'phase matched', i.e., $\mathbf{k}_p = 2\mathbf{k}$. For collinear modes, this is achieved by taking $\omega_p = 2\omega$. The spatial integral actually runs only over the region where the nonlinear index n_2 is different from zero. We also see from (1.137) that one 'pump photon' with energy $\hbar\omega_p = 2\hbar\omega$ can 'decay' into a pair of signal photons. We already anticipated this behaviour in the number state expansion (1.132).

1.3.4 Thermal states

This last class of field states is also quite important, but it is more general than the 'pure' states described before. In a thermal state, one has to use both classical and quantum statistics. The easiest way to deal with that is to use stationary states, hence the number states we found first. In this case, one can use Boltzmann statistics to describe the field at thermal equilibrium.

Each stationary state $|n\rangle$ can occur with a classical probability proportional to the Boltzmann factor $e^{-n\hbar\omega/k_BT}$. The normalization of these probabilities (the 'partition function') is a simple geometrical series

$$Z = \sum_{n=0}^{\infty} e^{-n\hbar\omega/k_BT} = \frac{1}{1 - e^{-\hbar\omega/k_BT}}, \tag{1.139}$$

so that the probability to find n photons in a single mode is given by

$$p_n(T) = (1 - e^{-\hbar\omega/k_BT})e^{-n\hbar\omega/k_BT} \tag{1.140}$$

[1]We are actually cheating with the polarization vector $\boldsymbol{\varepsilon}$. An accurate desription replaces n_2 by a third-rank tensor that produces a scalar out of three vectors.

At optical frequencies and room temperature, the exponential has a large negative argument for $n \geq 1$ so that the field is essentially at zero temperature. This is different for microwave radiation, e.g., or for star atmospheres.

Simple exercise: mean photon number. Result:

$$\langle\hat{n}\rangle_T = \frac{1}{e^{\hbar\omega/k_BT} - 1}. \tag{1.141}$$

Exercise: photon number variance. Result:

$$(\Delta n)_T^2 = \frac{e^{\hbar\omega/k_BT}}{(e^{\hbar\omega/k_BT} - 1)^2} = \frac{1}{4\sinh^2(\hbar\omega/2k_BT)}. \tag{1.142}$$

Electric field fluctuations in a single mode at finite temperature:

$$\langle\mathbf{E}^2(\mathbf{x}, t)\rangle_T = E_1^2 \sin^2 kz \left\langle a(t)a^\dagger(t) + a^\dagger(t)a(t)\right\rangle_T = E_1^2 \sin^2 kz\,(2\langle\hat{n}\rangle_T + 1) \tag{1.143}$$

they are enhanced by a factor $2\langle\hat{n}\rangle_T + 1 = \coth(\hbar\omega/2k_BT)$ compared to zero temperature.

Finally, the thermal state is often expressed in terms of a *density operator* $\hat{\rho}$. This is an operator on the space of number states. In thermal equilibrium, its expansion in the number state basis can be read off from the probabilities (1.140)

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n(T)|n\rangle\langle n| \tag{1.144}$$

Average values with respect to both quantum and classical statistics can now be obtained from a trace of the density operator. For the photon number operator, e.g.,

$$\langle\hat{n}\rangle_\rho = \sum_{n=0}^{\infty} n p_n(T) = \sum_{n=0}^{\infty}\langle n|\hat{n}\hat{\rho}|n\rangle = \mathrm{tr}\,(\hat{n}\hat{\rho}). \tag{1.145}$$

For an arbitrary operator A, this is generalized to

$$\langle\hat{A}\rangle_\rho = \mathrm{tr}\left(\hat{A}\hat{\rho}\right) = \sum_{n=0}^{\infty}\langle n|\hat{A}\hat{\rho}|n\rangle \tag{1.146}$$

The advantage of this formalism is that the trace can be taken in any basis, and one can choose a basis adapted to the operator whose average one its interested in. The second advantage of using the density matrix is that it gives a suitable description of a quantum system whose dynamics is not completely known and can only be specified by probabilities. In that case, one formulates an equation of motion for the density matrix from the solution of which the averages of all interesting quantities can be calculated.

1.4 Two modes, many modes

1.4.1 Multi-mode Hilbert space and observables

The state space of a two-mode field is the tensor product of the Fock spaces of two harmonic oscillators. In terms of number states, the basis vectors of this space can be written

$$|n_1; n_2\rangle = |n_1\rangle_{\text{mode}\,1} \otimes |n_2\rangle_{\text{mode}\,2}$$

where the first mode contains n_1 and the second mode n_2 photons. These states are called 'product states'. Expectation values of products of operators pertaining to mode 1 and 2 factorize, e.g.,

$$\langle \hat{n}_1 \hat{n}_2 \rangle = \langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle .$$

But due to the possibility of forming superpositions, there is much more 'space' in the multi-mode Hilbert space. For example, it is possible that two modes 'share' a single photon:

$$\frac{1}{\sqrt{2}} \left(|0; 1\rangle + |1; 0\rangle \right) \tag{1.147}$$

This state is called 'entangled' if no change of basis for the mode expansion exists such that the state is mapped onto a product state (this may be very difficult to check in practice).[2] The state is by no means unphysical, however, since it is generated by

$$\frac{1}{\sqrt{2}} (a_1^\dagger + a_2^\dagger) |0; 0\rangle \tag{1.148}$$

[2]It is simple to see, however, that the expectation value of $\hat{n}_1 \hat{n}_2$ does not factorize. Indeed, $\langle \hat{n}_1 \rangle = \frac{1}{2} = \langle \hat{n}_2 \rangle$ while $\langle \hat{n}_1 \hat{n}_2 \rangle = 0$ since in each component of the state (1.147), at least one mode has zero photons.

where $|0; 0\rangle$ is the two-mode vacuum. Such sums of creation operators occur always in the mode expansion of the quantized field. The decay of an excited atomic state, for example, generates a continuous superposition of one-photon states where an infinite number of modes share a single photon.

Many-mode single-photon states are also generated when an atom is illuminated by a single photon: the scattering of this photon by the atom generates, as in the classical electromagnetic theory, a continuous angular distribution of modes with a nonzero amplitude for one-photon excitations.

Finally, what about the density matrix for a multi-mode field? Let us start with the simple case of two modes of the same frequency in thermal equilibrium. According to the general rule, the density matrix is a sum of projectors onto the stationary states $|n_1; n_2\rangle$ of the two-mode system, each weighted with a probability proportional to $\mathrm{e}^{-\beta(n_1+n_2)}$. (Use $\beta = \hbar\omega / k_B T$.) Since the energy is made additively from single-mode energies, we can factorize this density operator:

$$\begin{aligned} \hat{\rho} &= Z^{-1} \sum_{n_1, n_2} \mathrm{e}^{-\beta(n_1+n_2)} |n_1; n_2\rangle\langle n_1; n_2| \\ &= Z \sum_{n_1} \mathrm{e}^{-\beta n_1} |n_1\rangle\langle n_1| \otimes \sum_{n_2} \mathrm{e}^{-\beta n_2} |n_2\rangle\langle n_2| \\ &= Z^{-1} \tilde{\rho}_1 \otimes \tilde{\rho}_2 \end{aligned} \tag{1.149}$$

where the $\tilde{\rho}_{1,2}$ are un-normalized density matrices. The tensor product of the projectors is defined by coming back to the tensor product of states

$$|n_1\rangle\langle n_1| \otimes |n_2\rangle\langle n_2| = \left(|n_1\rangle \otimes |n_2\rangle \right) \left(\langle n_1| \otimes \langle n_2| \right) .$$

The trace of the two-mode density matrix (1.149) also factorizes because the matrix elements of a tensor product operator are, by definition, the products of the individual matrix elements

$$\begin{aligned} \mathrm{tr}\,(\hat{\rho}) &= Z^{-1} \sum_{n_1, n_2} \langle n_1; n_2 | \tilde{\rho}_1 \otimes \tilde{\rho}_2 | n_1; n_2 \rangle \\ &= Z^{-1} \sum_{n_1, n_2} \langle n_1 | \tilde{\rho}_1 | n_1 \rangle \langle n_2 | \tilde{\rho}_2 | n_2 \rangle \\ &= Z^{-1} \left(\mathrm{tr}\, \tilde{\rho}_1 \right) \left(\mathrm{tr}\, \tilde{\rho}_2 \right) \end{aligned} \tag{1.150}$$

and therefore $Z = Z_1 Z_2 = (1 - \mathrm{e}^{-\beta})^{-2}$.

Since the density matrix of this thermal two-mode state factorizes, this state is not entangled (averages of products of single-mode operators factorize). This is no longer true, however, if we allow for an interaction between the modes. Then the energy is no longer a sum of single-mode energies, and the previous factorization does no longer work. This is by the way a general rule: interactions between quantum systems lead to entangled states. For this reason, entangled states are much more frequent in Nature than are factorized states. It is a nontrivial task, however, to decide whether a given density matrix describes an entangled state or not.

Digression (*Einschub*): tensor product states and operators

It is somewhat tricky to guess the right formulas for multimode field states and operators. The general rule is the following:

Field operator	$\leftrightarrow$	sum of modes
Field state	$\leftrightarrow$	product of modes

For example, the electric field operator for a two-mode field is given by

$$\mathbf{E}(\mathbf{x}, t) = E_1 \boldsymbol{\varepsilon}_1 a_1(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 a_2(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{h.c.}$$

while a typical state is for example the product state $|n_1; n_2\rangle = |n_1\rangle \otimes |n_2\rangle$. The general rule gets complicated (1) when we allow for superpositions (sums) of product states and (2) when we consider measurements that involve products of different mode operators.

In calculations, one often needs products of operators, like $\mathbf{E}^2(\mathbf{x}, t)$. These are computed in the usual way, one has just to take care that operators sometimes do not commute. But this is only relevant for operators acting on the same mode, $[a_1, a_1^\dagger] = 1$, while for different modes

$$[a_1, a_2^\dagger] = 0$$

because they correspond to independent degrees of freedom.

Operator averages in product states. Let us consider the average electric field for the two-mode case written above. Using the mode expansion,

we find terms like $\langle a_i(t)\rangle$ ($i = 1, 2$) and their adjoints. Now the operator $a_1|\psi\rangle$ is evaluated by letting a_1 act on the first factor of a product state:

$$a_1|n_1; n_2\rangle = (a_1|n_1\rangle) \otimes |n_2\rangle$$

If $|\psi\rangle$ is a sum of product states (entangled state), then this procedure is done for every term in this sum. Sometimes this is formalized by writing the operator as $a_1 \otimes \mathbb{1}$, thus indicating that for the second mode nothing happens. The action of such operator tensor products is apparently defined as

$$A_1 \otimes B_2 |n_1; n_2\rangle = A_1|n_1\rangle \otimes B_2|n_2\rangle \tag{1.151}$$

by letting each operator factor act on the respective state factor. This notation allows to avoid the subscripts 1 and 2 as the relevant mode is indicated by the position in the operator product.

Similarly, the scalar product of tensor products of states is defined by

$$\langle n_1; n_2 | m_1; m_2\rangle = \langle n_1| \otimes \langle n_2|m_1\rangle \otimes |m_2\rangle = \langle n_1|m_1\rangle\langle n_2|m_2\rangle$$

by taking the scalar product of the corresponding factors.

The average of the electric field for a product of number states is thus zero, as for a single-mode field, because $\langle n|an\rangle = 0$, and this is true for both modes. What about a product state of two coherent states, $|\psi\rangle = |\alpha; \beta\rangle$? It is simple to see that we get the classical result (we assume that both modes have the same frequency ω)

$$\langle \mathbf{E}(\mathbf{x}, t)\rangle = E_1 \boldsymbol{\varepsilon}_1 \alpha\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 \beta\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{c.c.} \tag{1.152}$$

(Note that 'c.c.' and not 'h.c.' occurs.) As a general rule, classical fields can be described by tensor products of coherent states.

Last example where we go quantum: a superposition of coherent product states,

$$|\psi\rangle = c|\alpha; \beta\rangle + d|\beta; \alpha\rangle$$

with some complex amplitudes c, d. Then we find

$$\langle a_1\rangle = |c|^2\alpha + |d|^2\beta$$

if $\langle\alpha|\beta\rangle = 0$. (This is actually never exactly the case, but can be achieved to a very good precision if $|\alpha - \beta| \gg 1$.) This result is an average over the two possible coherent amplitude, weighted with the corresponding probabilities. The average field thus becomes:

$$\langle \mathbf{E}(\mathbf{x},t)\rangle = E_1\boldsymbol{\varepsilon}_1\left(|c|^2\alpha + |d|^2\beta\right)\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2\boldsymbol{\varepsilon}_2\left(|c|^2\beta + |d|^2\alpha\right)\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{c.c.}$$

Question: this result does not allow to distinguish this state from an 'incoherent mixture' of coherent product states like in (1.152), each state occurring with a probability $|c|^2, |d|^2$. This mixture would be described by the density operator

$$\hat\rho_{\text{mix}} = |c|^2|\alpha;\beta\rangle\langle\alpha;\beta| + |d|^2|\beta;\alpha\rangle\langle\beta;\alpha|$$

and gives the same average electric field (exercise). If the coherent amplitudes α, β are closer together, then due to the nonzero overlap $\langle\alpha|\beta\rangle$, one can distinguish superposition and mixture (exercise). Are there observables that can make the difference in the case $\langle\alpha|\beta\rangle = 0$?

Average of single-mode operator. Let us calculate as another example the average photon number in mode 1 for a two-mode field in the entangled state (1.147). The relevant photon number operator is given by $a_1^\dagger a_1$ or, to be more precise, $a_1^\dagger a_1 \otimes \mathbb{1}$. Its action on the entangled state is worked out using linearity and the operator product rule (1.151)

$$\begin{aligned}
&\frac{1}{\sqrt2}a_1^\dagger a_1\otimes\mathbb{1}\left(|0;1\rangle + |1;0\rangle\right)\\
&= \frac{1}{\sqrt2}\left(a_1^\dagger a_1|0\rangle\otimes|1\rangle + a_1^\dagger a_1|1\rangle\otimes|0\rangle\right)\\
&= \frac{1}{\sqrt2}|1\rangle\otimes|0\rangle = \frac{1}{\sqrt2}|1;0\rangle
\end{aligned}$$

Taking the scalar product with the original state, we find

$$\langle\hat n_1\rangle = \frac12\left(\langle 0;1| + \langle 1;0|\right)|1;0\rangle = \frac12.$$

Once you have done this calculation, you can use the shorter rule: all we need are the probabilities of having $n_1 = 0, 1, \ldots$ photons in mode 1. For this, collect all product states in the state with the same number of photons n_1 and compute the squared norm of these states. From the probabilities for n_1 photons, you get the average photon number.

Product operators. As a second example, let us compute the average value of the product $a_i^\dagger a_j$ $(i,\, j = 1,\, 2)$ in a thermal two-mode state. This object occurs when you measure the two-mode field with a photodetector (see paragraph **??** below). The tensor product notation is more cumbersome here and gives

$$a_1^\dagger a_1\otimes\mathbb{1} \quad\text{or}\quad \mathbb{1}\otimes a_2^\dagger a_2 \quad\text{or}\quad a_1^\dagger\otimes a_2 \quad\text{or}\quad a_1\otimes a_2^\dagger.$$

The density matrix is a tensor product of thermal single-mode density matrices. We shall see that the result is:

$$\langle a_i^\dagger a_j\rangle_T = \delta_{ij}\bar n(T) \tag{1.153}$$

where $\bar n(T)$ is the average photon number in a single mode. How does this come about?

When $i = j$, we are left with the calculation of the average photon number for a single mode:

$$\langle a_i^\dagger a_i\rangle = \sum_{n_1,n_2}\langle n_1;n_2|a_i^\dagger a_i\hat\rho_1\otimes\hat\rho_2|n_1;n_2\rangle$$

The action of the product density operators factorizes:

$$\hat\rho_1\otimes\hat\rho_2|n_1;n_2\rangle = \hat\rho_1|n_1\rangle\otimes\hat\rho_2|n_2\rangle$$

Each single-mode density operator, acting on a number state, gives the corresponding occupation probability:

$$\hat\rho_1|n_1\rangle = \sum_{m_1}p_{m_1}(T)|m_1\rangle\langle m_1|n_1\rangle = p_{n_1}(T)|n_1\rangle,$$

so that we have, using the result for the photon number of one mode

$$\begin{aligned}
\langle a_i^\dagger a_i\rangle &= \sum_{n_1,n_2}p_{n_1}(T)p_{n_2}(T)\langle n_1;n_2|a_i^\dagger a_i|n_1;n_2\rangle\\
&= \sum_{n_1,n_2}p_{n_1}(T)p_{n_2}(T)n_i\\
&= \sum_{n_i}p_{n_i}(T)n_i\sum_{n_j}p_{n_j}(T)
\end{aligned}$$

In the last step, we have noted that the double sum can be factorized ($j \neq i$ is the other index). The second sum gives unity because the probabilities are normalized, the first sum gives the average photon number $\bar{n}(T)$ at temperature T and does no longer depend on the mode label (this is because we assumed equal frequencies for both modes). This completes the proof in the case $i = j$.

A similar calculation shows that the average of $a_1^\dagger a_2$ vanishes: indeed, we have

$$\langle n_1; n_2 | a_1^\dagger a_2 | n_1; n_2 \rangle = \langle n_1 | a_1^\dagger | n_1 \rangle \langle n_2 | a_2 | n_2 \rangle = 0.$$

1.4.2 Quantum theory of the beam splitter

A beamsplitter is the most simple way to mix two modes, see Figure 1.3. From classical electrodynamics, one gets the following amplitudes for the

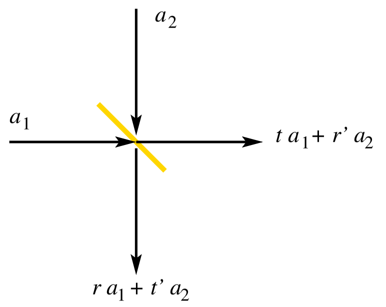

Figure 1.3: Mixing of two modes by a beam splitter.

outgoing modes:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}} \mapsto \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{out}} = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}}. \tag{1.154}$$

The recipe for quantization is now: 'replace the amplitudes by annihilation operators'. If the outgoing modes are still to be useful for the quantum theory, they have to satisfy the commutation relations:

$$\left[a_i(\text{out}),\, a_j^\dagger(\text{out})\right] = \delta_{ij}. \tag{1.155}$$

These conditions give constraints on the reflection and transmission amplitudes, for example $|t^2| + |r'^2| = 1$. (Note that this is not identical to energy conservation for the incoming mode $a_1(\text{in})$.)

We are now looking for a unitary operator that implements this beam-splitter transformation and from which we can also compute the transformation of the states. Let us start from the general transformation

$$a_i \mapsto A_i = S_{ij} a_j \qquad \text{or} \quad \vec{a} \mapsto \vec{A} = \mathsf{S}\,\vec{a} \tag{1.156}$$

where we have introduced matrix and vector notation. For the unitary transformation, we make the *ansatz*

$$U(\tau) = \exp\left(\mathrm{i}\tau B_{jk} a_j^\dagger a_k\right) \tag{1.157}$$

with B_{jk} a hermitean matrix (ensuring unitarity). The action of this unitary on the photon mode operators is now required to reduce to

$$a_i \mapsto A_i(\tau) \equiv U^\dagger(\tau) a_i U(\tau) \stackrel{!}{=} S_{ij} a_j. \tag{1.158}$$

We compute this 'conjugated operator' with the usual trick via a differential equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} A_i(\tau) &= -\mathrm{i} B_{jk} U^\dagger(\tau) \left[a_j^\dagger a_k,\, a_i\right] U(\tau) && (1.159)\\ &= -\mathrm{i} B_{jk} U^\dagger(\tau) \left(-\delta_{ij} a_k\right) U(\tau) && (1.160)\\ &= \mathrm{i} B_{ik} A_k(\tau). && (1.161) \end{aligned}$$

This is a system of linear differential equations with constant coefficients, so that we get as solution

$$\vec{A}(\tau) = \exp\left(\mathrm{i}\tau \mathsf{B}\right) \vec{A}(0) = \exp\left(\mathrm{i}\tau \mathsf{B}\right) \vec{a}. \tag{1.162}$$

We thus conclude that the matrix B is fixed by

$$\mathsf{S} = \exp\left(\mathrm{i}\tau \mathsf{B}\right). \tag{1.163}$$

If the transformation is part of a continuous group, we can expand it around unity. Doing the same for the matrix exponential, we get

$$\begin{aligned} \mathsf{S} &\approx \mathbb{1} + \mathrm{i}\epsilon \mathsf{T} + \ldots && (1.164)\\ \exp\left(\mathrm{i}\tau \mathsf{B}\right) &\approx \mathbb{1} + \mathrm{i}\tau \mathsf{B} + \ldots && (1.165)\\ \tau \mathsf{B} &= \epsilon \mathsf{T}. && (1.166) \end{aligned}$$

The unitary transformation is thus determined via the *generator* of the mode transformation. More precisely, the unitary $U(\epsilon) = \exp(\mathrm{i}\epsilon B_{jk} a_j^\dagger a_k)$ implements the 'one-parameter subgroup' of mode transformations $\mathsf{S}(\epsilon)$ that is the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathsf{S}(\epsilon) = \mathrm{i}\mathsf{T}\mathsf{S}(\epsilon).$$

A formal solution can be written as $\mathsf{S}(\epsilon) = \exp(\mathrm{i}\epsilon\mathsf{T})\mathsf{S}(0)$ so that with Eq.(1.163), we find indeed $\tau\mathsf{B} = \epsilon\mathsf{T}$.

Example: splitting a single photon state. For the beam splitter, an admissible transformation is given by

$$\mathsf{S} = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{1.167}$$

It generator is given by, for small θ,

$$T = \theta \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \theta\sigma_2 \tag{1.168}$$

so that the corresponding unitary operator reads

$$U(\tau) = \exp\left[\mathrm{i}\tau(-\mathrm{i}a_1^\dagger a_2 + \mathrm{i}a_2^\dagger a_1)\right] = \exp\left[\tau(a_1^\dagger a_2 - a_2^\dagger a_1)\right]. \tag{1.169}$$

The value of the parameter τ can be fixed via the identity

$$\exp(\mathrm{i}\tau\sigma_2) = \cos\tau + \mathrm{i}\sigma_2\sin\tau = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}, \tag{1.170}$$

so that $\tau = \theta$ for a general (not infinitesimal) transformation.

Finally, compute the state of the two-mode system if one photon is incident in mode 1 on the beam splitter. The final state is then, for small θ

$$\begin{aligned} |\psi\rangle &= U(\theta)|1,0\rangle \approx |1,0\rangle + \theta(a_1^\dagger a_2 - a_2^\dagger a_1)|1,0\rangle && (1.171) \\ &= |1,0\rangle - \theta|0,1\rangle. && (1.172) \end{aligned}$$

For finite θ, the higher powers also contribute. The calculation gets easy with the beam splitter transformation of the creation operators.

$$\begin{aligned} |\psi\rangle &= U(\theta)a_1^\dagger|0,0\rangle && (1.173) \\ &\stackrel{(1)}{=} U(\theta)a_1^\dagger U^\dagger(\theta)|0,0\rangle && (1.174) \\ &\stackrel{(2)}{=} \left(a_1^\dagger\cos\theta - a_2^\dagger\sin\theta\right)|0,0\rangle && (1.175) \\ &= \cos\theta|1,0\rangle - \sin\theta|0,1\rangle && (1.176) \end{aligned}$$

In step (1), we have used that the unitary operator leaves the vacuum state unchanged. (This is because we have written the exponent in normal order.) In step (2), we have used that $U^\dagger(\theta)$ implements the transformation inverse to $U(\theta)$. Re-introducing the transmission amplitudes, we find

$$|1,0\rangle \mapsto t|1,0\rangle + r|0,1\rangle \tag{1.177}$$

so that the probability amplitudes to find the photon in either output mode correspond exactly, for this incident one-photon state, to the classical transmission and reflection amplitudes.